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Concept Of The Month
Equilateral Triangle

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CHAPTER-WISE PREVIOUS YEAR KVPY Qs

OCTOBER 2018

MATHEMATICS TIMES

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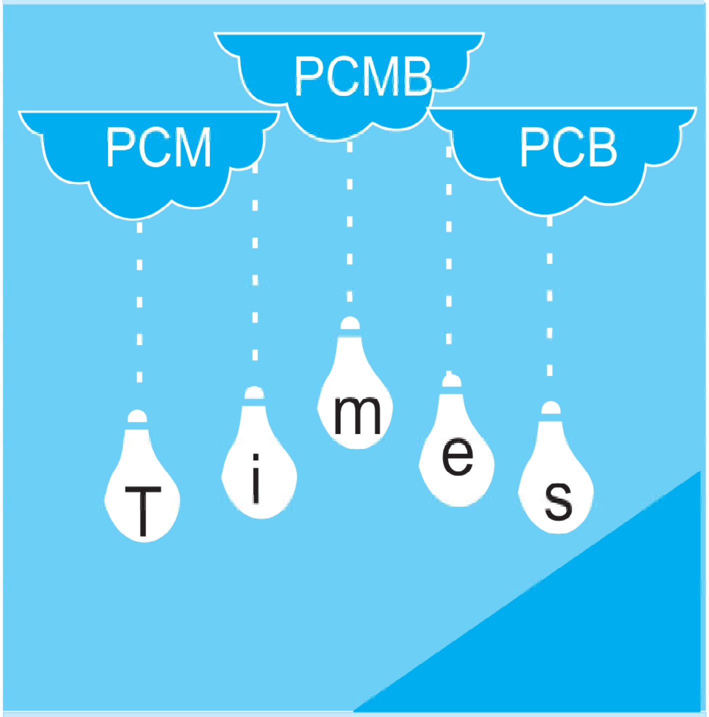
VOL-IV



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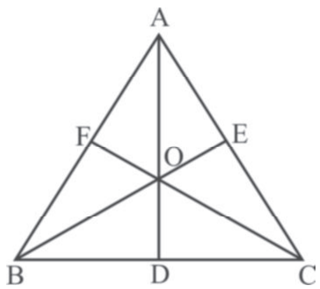
Equilateral Triangle

Concept of the month

This column is aimed at preparing students for all competitive exams like JEE, BITSAT etc. Every concept has been designed by highly qualified faculty to cater to the needs of the students by discussing the most complicated and confusing concepts in Mathematics.

By. DHANANJAYA REDDY THANAKANTI
(Bangalore)

An **equilateral triangle** is a triangle whose three sides all have the same length. They are the only **regular polygon** with three sides, and appear in a variety of contexts, in both **basic geometry** and more advanced topics such as complex number geometry and geometric inequalities.

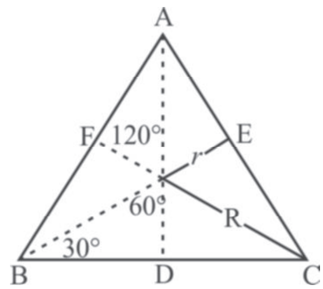


Identification

The most straight forward way to identify an equilateral triangle is by comparing the side lengths. If the three side lengths are equal, the structure of the triangle is determined (a consequence of SSS congruence). However, this is not always possible.

Another useful criterion is that the three angle of an equilateral triangle are equal as well, and are thus each $60°$. Since the angles opposite equal sides are themselves equal, this means discovering two equal angles of $60°$.

Notably, the equilateral triangle is the unique polygon for which the knowledge of only one side length allows one to determine the full structure of the polygon. For example, there are infintely many quadrilaterals with equal side lengths (rhombus) so we need to know at least one more property to determine its full structure. In this way, the equilateral triangle is in company with the circle and the sphere whose full structure are determined by supplying only the radius.



Basic Properties:

Because the equilateral triangle is, in some sense, the simplest polygon, many typically important property are easily calculable. For instance, for an equilateral triangle with side length a, we have the following :

- The altitude, median, angle bisector, and perpendicular bisector for each side are all the same single line.
- These 3 lines (one for each side) are also the **lines of symmetry** of the triangle.
- All three of the lines mentioned above have the same length of $\frac{a\sqrt{3}}{2}$.
- The area of an equilateral triangle is $\frac{a^2\sqrt{3}}{4}$.
- The **orthocenter**, **circumcenter**, **incenter**, **centroid** and **nine-point center** are all the same point. The Euler line degenerates into a single point.
- The circumradius of an equilateral triangle is $\frac{a\sqrt{3}}{3}$.

  Note that this is $\frac{2}{3}$ the length of an altitude,

  because each altitude is also a median of the triangle.
- The **inradius** of an equilateral triangle is $\frac{a\sqrt{3}}{6}$.

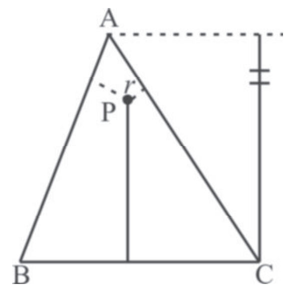
  Note that the inradius is $\frac{1}{3}$ the length of an altitude, because each altitude is also a median of the triangle. Also the inradius is $\frac{1}{2}$ the length of a circumradius.

  It is also worth nothing that six congruent equilateral triangles can be arranged to form a regular hexagon, making several properties of regular hexagons easily discoverable as well. For example, the area of a regular hexagon with side length a is simply 6. $\frac{a^2\sqrt{3}}{4} = \frac{3a^2\sqrt{3}}{2}$.

Advanced Properties

Firstly, it is worth noting that the circumradius is exactly twice the inradius, which is important as $R \geq 2r$ according to Euler's inequality. The equilateral triangle provides the equality case, as it does in more advanced cases such as the **Erdos-Mordell inequality**.

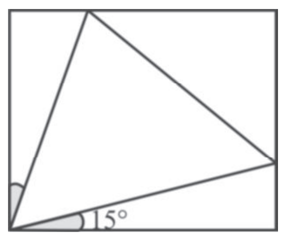
If P is any point inside an equilateral triangle, the sum of its distances from three sides is equal to the length of an altitude of the triangle:



Equilateral Triangle

The equilateral triangle is also the only triangle that can have both rational side lengths and angles (when measured in degrees).

When inscribed in a unit square, the maximal possible area of an equilateral triangle is $2\sqrt{3}-3$, occurring when the triangle is oriented at a $15°$ angle and has sides of length $\sqrt{6}-\sqrt{2}$:



It is also worth noting that besides the equilateral triangle in the above picture, there are three other triangles with areas X, Y, and Z (with Z the largest): They satisfy the relation $2X = 2Y = Z$ $\Rightarrow X + Y = Z$. in fact, $X + Y = Z$ is true of any rectangle circumscribed about an equilateral triangle, regardless of orientation.

Equilateral triangles are particularly useful in the complex plane, as their vertices a, b, c satisfy the relation

$$a + b\omega + c\omega^2 = 0,$$

where ω is a primitive third root of unity, meaning $\omega^3 = 1$ and $\omega \neq 1$. In particular, this allows for an easy way to determine the location of the final vertex, given the locations of the remaining two.

Another property of the equilateral triangle is Van Schooten's theorem:

Theorem: If ABC is an equilateral triangle and M is a point on the arc BC of the circumcircle of the triangle ABC, then

$$MA = MB + MC.$$

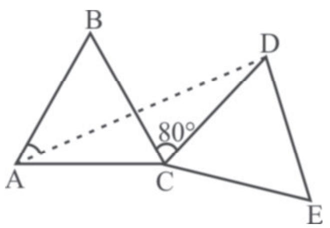
Proof: Using the Ptolemy's theorem on the cyclic quadrilateral $ABMC$, we have

$$MA \cdot BC = MB \cdot AC + MC \cdot AB$$

or $$MA = MB + MC$$

Exercise

1. Let ΔABC and ΔCDE be equilateral triangles of the same size, and $\angle BCD = 80°$ between them. Find the measure of $\angle BAD$ in degrees.



(a) $40°$ (b) $60°$
(c) $30°$ (d) None of these

2. A triangle has a $60°$ angle. If two of its sides are 1 *m* long, how many different triangles can one draw which fit these measurements?

(a) 1 (b) 2
(c) 3 (d) Infinitely many

3. A square and an equilateral triangle have the same perimeter. If the area of the triangle is $16\sqrt{3}$, what is the area of the square ?

(a) 16 (b) 36
(c) 40 (d) None of these

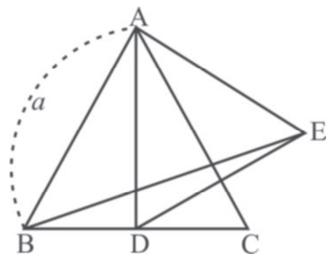
4. Given two distinct points A and B in the plane, how many distinct points C are there on the same plane such that ΔABC is an equilateral triangle?

(a) 1 (b) 2
(c) 3 (d) infinitely many

5. There are 8 equilateral triangles each of which is 4 cm on a side. All of these triangles have been made by bending copper wires. Now, you unbend the wires and try to make squares with side length 1 cm: How many such squares can you make?

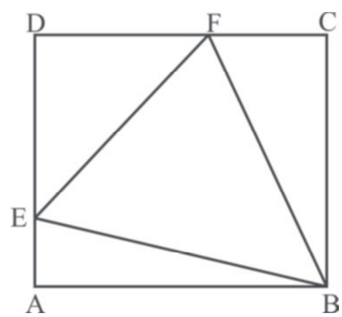
(a) 22 (b) 24 (c) 25 (d) 26

6. In the below diagram, the side length of equilateral triangle ΔABC is $a = 3$. If D is the midpoint of $\overline{BC}$ and ΔADE is also an equilateral triangle, what is the area of ΔABE?



(a) $\frac{11\sqrt{3}}{4}$ (b) $\frac{7\sqrt{3}}{2}$ (c) $\frac{9\sqrt{3}}{4}$ (d) $\frac{13\sqrt{3}}{2}$

7. Points E and F are located on square $ABCD$ so that ΔBEF is equilateral. What is the ratio of the area of ΔDEF to that ΔABE?



(a) $\frac{4}{3}$ (b) $\frac{3}{2}$ (c) $\sqrt{3}$ (d) 2

8. Show that there is no equilateral triangle in the plane whose vertices have integer coordinates.

ANSWER KEY

1. a **2.** a **3.** b **4.** b **5.** b

6. c **7.** d

HINTS & SOLUTIONS

1.Sol: Consider isosceles $\Delta ACD.$

$\angle ACD = 60 + 80 = 140°$

Since

$AC = CD, \angle ADC = \angle DAC = \frac{180° - 140°}{2} = 20°$

It follows that $\angle \alpha = 40°.$

2.Sol: Take $60°$ as vertex angle. Other two angles

$$= \frac{180° - 60°}{2} = 60°.$$

Take $60°$ as base angle. Vertex angle

$$= 180° - (60° + 60°) = 60°.$$

In either of the cases the triangle will be an equilateral one. Hence only 1 triangle .

3.Sol: Let a be the side of the triangle and b the side of the square. We know that $3a = 4b.$

Now, the area of the triangle is given by $\frac{a^2\sqrt{3}}{4}$, so

$\frac{a^2\sqrt{3}}{4} = 16\sqrt{3} \Rightarrow \frac{a^2}{4} = 16 \Rightarrow a^2 = 64.$

Hence $a = 8.$

We have: $3(8) = 4b \Rightarrow b = \frac{3(8)}{4} = 6.$

Finally, the area of the square is $6^2 = 36.$

4.Sol: We know AB is a fixed line. given that ΔABC is equilateral, that is we need to draw two intersecting circles of radius of length AB at A, B as centers of respective circles. Therefore there are two intersecting points. That is two distincts points are possible for the point 'C'.

5.Sol: We know length of the copper wire is 8 times the perimeter of equilateral triangle. That is

$8(3\times 4)\,cm.$

Now, we have length of copper wire is 96 cm.
We know perimeter of a Square is 4(1) cm.

$\therefore$ total number of squares can be made using 96 cm copper wire is $\frac{96}{4} = 24$.

6.Sol: We know AD is altitude of equilateral triangle with side 3

$$\therefore \qquad AD = \frac{3\sqrt{3}}{2}$$

we also given ΔADE is equilateral, that is $AD = AE = \frac{3\sqrt{3}}{2}$

now $\angle BAC + \angle CAE = 90°$

$\therefore \quad \Delta BAE$ is right angled triangle.

Hence area of $\Delta ABE = \frac{1}{2}AE \times AB = \frac{9\sqrt{3}}{4}$

7.Sol: Since triangle BEF is equilateral, $EA = FC$, and EAB and FCB are SAS congruent. Thus, triangle DEF is an isosceles right triangle. So we let $DE = x.$ Thus $EF = EB = FB = x\sqrt{2}.$ If we go angle chasing, we find out that $\angle AEB = 75°$, thus $\angle ABE = 15°.$ $\frac{AE}{EB} = \sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}.$ Thus $\frac{AE}{x\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$, or $AE = \frac{x(\sqrt{3}-1)}{2}.$ Thus $AB = \frac{x(\sqrt{3}+1)}{2}$, and $[ABE] = \frac{x^2}{4}$, and $[DEF] = \frac{x^2}{2}.$ Thus the ratio of the area is (d) 2.

Method 2 : (Non - trig) : Let the side length of $ABCD$ be 1. Let $DE = x.$ It suffices that $AE = 1 - x.$ Then triangles ABE and CBF are congruent by HL, so $CF = AE$ and $DE = DF.$

We find that $BE = EF = x\sqrt{2}$, and so, by the Pythagorean Theorem, we have $(1-x)^2 + 1 = 2x^2.$

Thus yields $x^2 + 2x = 2$, so $x^2 = 2 - 2x.$ Thus, the desired ratio of area is

$$\frac{\frac{x^2}{2}}{\frac{1-x}{2}} = \frac{x^2}{1-x} = 2$$

Method 3 : ΔBEF is equilateral, so $\angle EBF = 60°$, and $\angle EBA = \angle FBC$ so they must each so they must each be $15°$. Then let $BE = EF = FB = 1$, which gives $EA = \sin 15°$ and $AB = \cos 15°$. The area of ΔABE is then $\frac{1}{2}\sin 15° \cos 15° = \frac{1}{4}\sin 30° = \frac{1}{8}$.

ΔDEF is an isosceles right triangle with hypotenuse 1, so $DE = DF = \frac{1}{\sqrt{2}}$ and therefore its area is $\frac{1}{2}\left(\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) = \frac{1}{4}$. The ratio of areas is

then $\dfrac{\frac{1}{4}}{\frac{1}{8}} = 2$

8.Sol: Suppose that there is an equilateral triangle in the plane whose vertices have integer coordinates.
The determinant formula for area is rational, so if the all three points are rational points, then the area of the triangle is also rational.

On the other hand, the area of an equilateral triangle with side length a is $\frac{a^2\sqrt{3}}{4}$, which is irrational

since a^2 is an integer and $\sqrt{3}$ is an irrational number.

This is a contradiction.

RECREATIONAL MATHS

Problem:

Suppose you have a white box with 60 white balls and a black box with 60 black balls (Fig). You take 20 balls from the white box, put them into the black box, and mix everything up thoroughly. Now you take 20 balls (most likely some white, some black) from the black box and put them into the white box. In the end, which is larger: the number of black balls in the white box or the number of white balls in the black box?

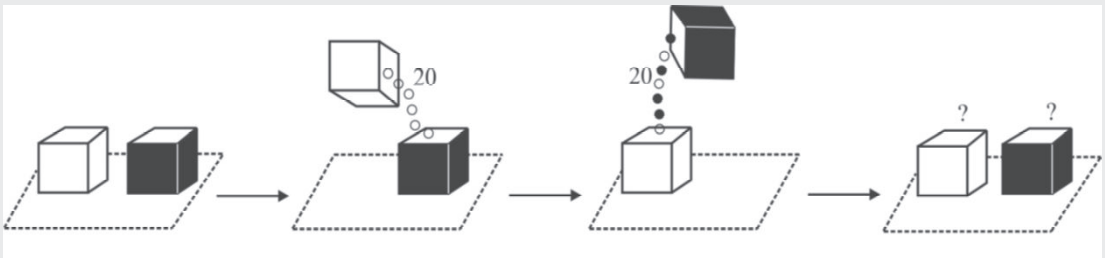


Fig: White and black balls

Solution to the above problem will be published in the next month issue.

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CONICS

1. If the focus of the parabola $(y-\beta)^2 = 4(x-\alpha)$ always lies between the lines $x+y=1$ and $x+y=3$, then

(a) $1<\alpha+\beta<2$ (b) $0<\alpha+\beta<1$

(c) $0<\alpha+\beta<2$ (d) None of these

2. The tangents at two points P and Q on the parabola $y^2 = 4x$ intersect at T. If SP, ST and SQ are equal to a, b an c respectively, where S is the focus, then the roots of the equation $ax^2+2bx+c=0$ are

(a) Real and equal (b) Real and unequal

(c) Complex numbers (d) Irrational

3. $ABCD$ and $EFGC$ are squares and the curve $y=k\sqrt{x}$ passes through the origin D and the points B and F. The ratio $\frac{FG}{BC}$ is :

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{\sqrt{3}+1}{2}$

(c) $\frac{\sqrt{5}+1}{4}$ (d) $\frac{\sqrt{3}+1}{4}$

4. In a square matrix A of order 3, $a_{ii} = m_i + i$ where $i = 1, 2, 3$ and m_i's are the slopes (in increasing order of their absolute value) of the 3 normals concurrent at the point $(9,-6)$ to the parabola $y^2 = 4x$. Rest all other entries of the matrix are one. The value of det.(A) is equal to :

(a) 37 (b) –6 (c) –4 (d) –9

5. Through the vertex O of a parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. The locus of the middle point of PQ is

(a) $y^2 = 2x+8$ (b) $y^2 = x+8$

(c) $y^2 = 2x-8$ (d) None of these

6. If the two parabolas $y^2 = 4a(x-k_1)$ and $x^2 = 4a(y-k_2)$ always touch each other, k_1 and k_2 being variable parameters, then their point of contact lies on the curve

(a) $xy = a^2$ (b) $xy = 2a^2$

(c) $xy = 4a^2$ (d) None of these

7. If $\frac{x^2}{f(4a)} + \frac{y^2}{f(a^2-5)} = 1$ represents an ellipse with major axis as y-axis and f is a decreasing function, such that $f(x) > 0, \forall x \in R$, then complete set of values of a is :

(a) $(-\infty, 1)$ (b) $(-\infty,-1)\cup(5,\infty)$

(c) $(-1,4)$ (d) $(-1,5)$

8. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding point Q on the auxilary circle meet on the line :

(a) $x = a/e$ (b) $x = 0$

(c) $y = 0$ (d) None of these

9. The locus of mid-points of focal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{ex}{a}$ (b) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{ex}{a}$

(c) $x^2+y^2=a^2+b^2$ (d) None of these

10. The maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis is

(a) $\sqrt{3}ab$ (b) $\frac{3\sqrt{3}}{4}ab$

(c) $\frac{5\sqrt{3}}{4}ab$ (d) None of these

11. A tangent to the ellipse $x^2+4y^2=4$ meets the ellipse $x^2+2y^2=6$ at P and Q. The angle between the tangent at P and Q of the ellipse $x^2+2y^2=6$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

12. The normal at a variable point P on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of eccentricity e meets the axes of the ellipse in Q and R then the locus of the mid-point of QR is a conic with an eccentricity e' such that :

(a) e' is independent of e (b) $e'=1$

(c) $e'=e$ (d) $e'=1/e$

13. If normal at any point P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ meet the axes at M and N so that $\frac{PM}{PN}=\frac{2}{3}$, then the value of eccentricity is :

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{2}}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) None of these

14. The area of the triangle formed by the line $x-y=0, x+y=0$ and any tangent to the hyperbola $x^2-y^2=a^2$ is

(a) $2a^2$ (b) $4a^2$

(c) a^2 (d) None of these

15. A conic passes through the point (2,4) and is such that the segment of any of its tangents at any point contained between the co-ordinate is bisected at the point of tangency. Then the foci of the conic are:

(a) $(2\sqrt{2},0)$ and $(-2\sqrt{2},0)$

(b) $(2\sqrt{2},2\sqrt{2})$ and $(-2\sqrt{2},-2\sqrt{2})$

(c) (4, 4) and (-4, -4)

(d) $(4\sqrt{2},4\sqrt{2})$ and $(-4\sqrt{2},-4\sqrt{2})$

16. With one focus of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is:

(a) Less than 2 (b) 2

(c) $\frac{11}{3}$ (d) None of these

17. If a ray of light incident along the line $3x+\left(5-4\sqrt{2}\right)y=15,$ gets reflected from the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ then its reflected ray goes along the line

(a) $x\sqrt{2}-y+5=0$ (b) $\sqrt{2}y-x+5=0$

(c) $\sqrt{2}y-x-5=0$ (d) None of these

18. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h,k) is the point of intersection of the normals at P and Q, then k is equal to

(a) $\frac{a^2+b^2}{a}$ (b) $-\left(\frac{a^2+b^2}{a}\right)$

(c) $\frac{a^2+b^2}{b}$ (d) $-\left(\frac{a^2+b^2}{b}\right)$

19. All the chords of the hyperbola $3x^2-y^2-2x+4y=0$, subtending a right angle at the origin pass through the fixed point

(a) (1,-2) (b) (-1,2)

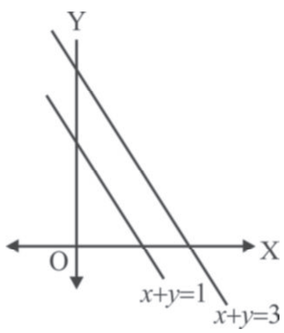
(c) (1,2) (d) None of these

ANSWER KEY

1. c	**2.** a	**3.** a	**4.** c	**5.** c
6. c	**7.** d	**8.** c	**9.** a	**10.** b
11. a	**12.** c	**13.** c	**14.** c	**15.** c
16. b	**17.** c	**18.** d	**19.** a	

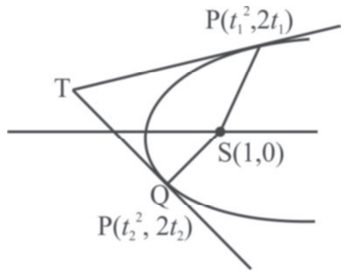
HINTS & SOLUTIONS

1.Sol: The coordinates of the focus of the given parabola are $(\alpha+1,\beta)$.



Clearly, focus must lie to the opposite side of the origin w.r.t. the line $x+y-1=0$ and same side as origin with respect to the line $x+y-3=0$. Hence, $\alpha+\beta>0$ and $\alpha+\beta<2$.

2.Sol: The tangents at the points $P(t_1^2,2t_1)$ and $Q(t_2^2,2t_2)$ intersect at the point $T(t_1t_2,t_1+t_2)$



Now, $a=SP=1+t_1^2$ and $c=SQ=1+t_2^2$

$\therefore$ $b^2=ST^2=(t_1t_2-1)^2+(t_1+t_2)^2$

$=t_1^2+t_2^2+1+t_1^2t_2^2$

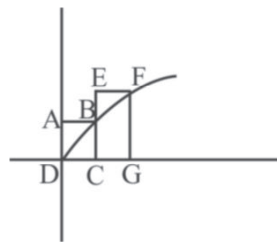
$=(1+t_1^2)(1+t_2^2)=ac$

$\therefore$ Roots of the equation $ax^2+2bx+c=0$ are real and equal.

3.Sol: $y^2=k^2x \Rightarrow y^2=4\left(\frac{k^2}{4}\right)x$

Given that B, F are on the curve

i.e., $B\left(\frac{k^2}{4}t_1^2,\frac{k^2}{2}t_1\right),F\left(\frac{k^2}{4}t_2^2,\frac{k^2}{2}t_2\right)$



also given that $ABCD$, $EFGC$ are square

i.e., $CD=BC$ and $CG=FG$

$\Rightarrow \frac{k^2}{4}t_1^2=\frac{k^2}{2}t_1 \Rightarrow t_1=2$ and

$\frac{k^2}{2}t_2=\frac{k^2}{4}(t_2^2-t_1^2)$

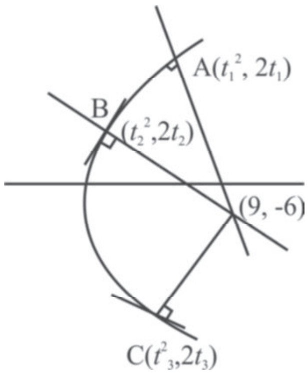
i.e., $2t_2=t_2^2-4 \Rightarrow t_2^2-2t_2-4=0$

$\Rightarrow$ $t_2=1+\sqrt{5}$

$$\frac{FG}{BC}=\frac{t_2}{t_1}=\frac{1+\sqrt{5}}{2}$$

4.Sol: Equation of normal at $(t^2, 2t)$

i.e., $\quad y=-tx+2t+t^3$



Given that, 3 normals are concurrent at $(9,-6)$

i.e., $\quad t^3-7t+6=0$

$\Rightarrow \quad (t-1)(t-2)(t+3)=0$

$\Rightarrow \quad t=1,2-3$

now $\quad a_{11}=m_1+1=-1+1=0;$

$a_{22}=m_2+2=-2+2=0;$

$a_{33}=m_3+3=3+3=6;$

$$\therefore \quad |A|=\begin{vmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 6\end{vmatrix}=-4$$

5.Sol: Given parabola is $y^2=4x$ (1)

$\Rightarrow \quad a=1$

Let $\quad P\equiv(t_1^2, 2t_1)$ and $Q\equiv(t_2^2, 2t_2)$

Slope of $OP=\frac{2t_1}{t_1^2}=\frac{2}{t_1}$ and slope of $OQ=\frac{2}{t_2}$

Given that OP is right angled to OQ

i.e., $OP\perp OQ$, $\therefore \frac{4}{t_1t_2}=-1$ or $t_1t_2=-4$ (2)

Let $R(\alpha,\beta)$ be the middle point of PQ, then

$$\alpha=\frac{t_1^2+t_2^2}{2} \quad (3)$$

$$\beta=t_1+t_2 \quad (4)$$

From (4), $\beta^2=t_1^2+t_2^2+2t_1t_2=2\alpha-8$

[From (2) and (3)]

Hence locus of $R(\alpha,\beta)$ is $y^2=2x-8$.

6.Sol: Given parabolas are $y^2=4a(x-k_1)$ (1)

and $\quad x^2=4a(y-k_2)$ (2)

Equation of tangent to (1) at (α,β) is

$$\beta y=2a(x-k_1+\alpha)$$

$\Rightarrow \quad 2ax-\beta y=2a(k_1-\alpha)$ (3)

Equation of tangent to (2) at (α,β) is

$$\alpha x=2a(y-k_2+\beta)$$

$\Rightarrow \quad \alpha x-2ay=2a(\beta-k_2)$ (4)

Since (3) and (4) are identical, comparing coefficients of x and y in (3) and (4), we get

$$\frac{2a}{\alpha}=\frac{\beta}{2a}$$

$\Rightarrow \alpha\beta=4a^2$. i.e., the point of contact (α,β) lies on the curve $xy=4a^2$.

7.Sol: We have $0<f(4a)<f(a^2-5)$

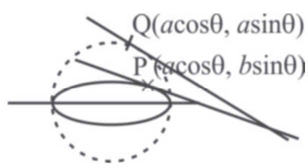
$4a>a^2-5$

$\Rightarrow \quad (a-5)(a+1)<0$

$\Rightarrow \quad a\in(-1,5)$

8.Sol: Equation of tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \quad (1)$$



Equation of tangent at Q is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{a}=1 \quad (2)$$

From $(1)-(2)$

$$\Rightarrow \quad y\sin\theta\left(\frac{1}{b}-\frac{1}{a}\right)=0$$

$$\Rightarrow \quad y=0$$

$\therefore$ Point of intersection lie on line $y=0$

9.Sol: Let (h,k) be the mid point of a focal chord.

Then its equation is $T=S_1$

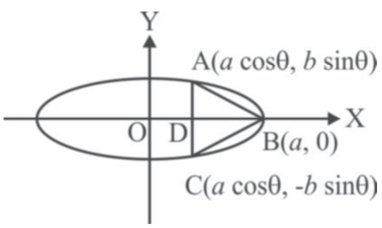
i.e., $$\frac{xh}{a^2}+\frac{ky}{b^2}-1=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$$

Since it passes through $(ae,0)$

$$\therefore \quad \frac{hae}{a^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$\therefore$ Locus of (h,k) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{xe}{a}$

10.Sol: The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$



Let $\quad A\equiv(a\cos\theta,-b\sin\theta)$

Then, $\quad C\equiv(a\cos\theta,-b\sin\theta)$

$$\Delta=\text{Area of } \Delta ABC=\frac{1}{2}\times AC\times BD=AD\times BD$$

$$=b\sin\theta(a-a\cos\theta)$$

$$=\frac{1}{2}ab(2\sin\theta-\sin 2\theta)$$

Now, $$\frac{d\Delta}{d\theta}=\frac{1}{2}ab(2\cos\theta-2\cos 2\theta)=0$$

$$\Rightarrow \quad \cos 2\theta=\cos\theta\Rightarrow 2\cos^2\theta-\cos\theta-1=0$$

$$\Rightarrow \quad (2\cos\theta+1)(\cos\theta-1)=0$$

$$\Rightarrow \quad \cos\theta=\frac{-1}{2} \text{ or } \cos\theta=1$$

If $\theta=0, \Delta=0$, which is not possible.

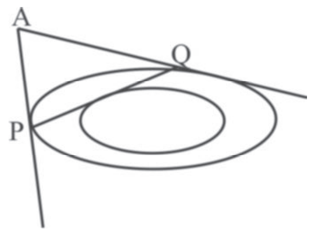
$$\therefore \quad \theta=2\pi/3.$$

$$\therefore \Delta_{\max}=\frac{3\sqrt{3}}{4}ab$$

11.Sol: We can write the ellipse $x^2+4y^2=4$ as

$$\frac{x^2}{4}+y^2=1 \qquad (1)$$

Equation of any tangent to the ellipse (1) can be written as $\frac{x}{2}\cos\theta+y\sin\theta=1 \qquad (2)$



Equation of the second ellipse can be written as

$$\frac{x^2}{6}+\frac{y^2}{3}=1 \qquad (3)$$

Suppose, the tangents at P and Q meet in $A(h,k)$. Equation of the chord of contact of the tangents through $A(h,k)$ is

$$\frac{hx}{6}+\frac{ky}{3}=1 \qquad (4)$$

Since (4) and (2) represent the same line

$$\therefore \quad \frac{h/6}{\frac{\cos\theta}{2}}=\frac{k/3}{\sin\theta}=\frac{1}{1}$$

$$\Rightarrow \quad h=3\cos\theta \text{ and } k=3\sin\theta.$$

Thus, coordinates of A are $(3\cos\theta, 3\sin\theta)$.
The joint equation of the tangents at A is given by $T^2=SS_1$

i.e., $$\left(\frac{hx}{6}+\frac{ky}{3}-1\right)^2=\left(\frac{x^2}{6}+\frac{y^2}{3}-1\right)\left(\frac{h^2}{6}+\frac{k^2}{3}-1\right) \qquad (5)$$

Let $a=$ coefficient of x^2 in (5)

$$=\frac{h^2}{36}-\frac{1}{6}\left(\frac{h^2}{6}+\frac{k^3}{3}-1\right)=\frac{-k^2}{18}+\frac{1}{6}$$

and, $b =$ coefficient of y^2 in (5)

$$= \frac{-k^2}{9} - \frac{1}{3}\left(\frac{h^2}{6} + \frac{k^3}{3} - 1\right) = \frac{-h^2}{18} + \frac{1}{3}.$$

We have, $a + b = \frac{-1}{18}\left(h^2 + k^2\right) + \frac{1}{6} + \frac{1}{3}$

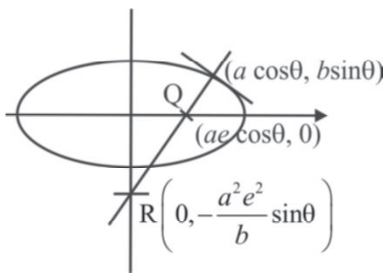
$$= \frac{-1}{18}\left(9\cos^2\theta + 9\sin^2\theta\right) + \frac{1}{2}$$

$$= \frac{-1}{18}(9) + \frac{1}{2} = 0.$$

Thus, (5) represents two lines which are at right angles to each other.

12.Sol: Let (h,k) be the variable point P on ellipse

i.e., $2h = ae^2\cos\theta, 2k = \frac{-a^2e^2\sin\theta}{b}$



$$\Rightarrow \quad \frac{4h^2}{\left(ae^2\right)^2} + \frac{4k^2}{\left(a^2\frac{e^2}{b}\right)^2} = 1$$

$$\frac{x^2}{\left(\frac{ae^2}{2}\right)^2} + \frac{y^2}{\left(\frac{a^2e^4}{2b}\right)} = 1$$

$$\frac{a^2e^4}{4} = \frac{a^4e^4}{4b^2}\left(1 - (e')^2\right)$$

$$\Rightarrow \quad b^2 = a^2\left(1 - (e')^2\right)$$

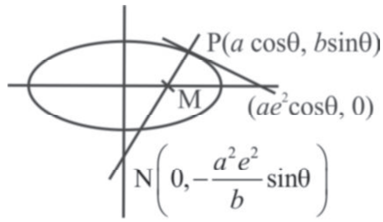
$$\Rightarrow \quad e = e'$$

13.Sol: $(PM)^2 = a^2\cos^2\theta\left(1 - e^2\right)^2 + b^2\sin^2\theta$

$$= b^2\left(1 - e^2\cos^2\theta\right)$$

$$(PN)^2 = a^2\cos^2\theta + \frac{\sin^2\theta}{b^2}\left(b^2 + a^2e^2\right)^2$$

$$= \frac{a^4}{b^2}\left(1 - e^2\cos^2\theta\right)$$



$$\left(\frac{PM}{PN}\right)^2 = \frac{b^4}{a^4} = \frac{4}{9}$$

$$\frac{b^2}{a^2} = \frac{2}{3} = 1 - e^2 \Rightarrow e = \frac{1}{\sqrt{3}}$$

14.Sol: Any tangent at $P(a\sec\theta, b\tan\theta)$ to the hyperbola $x^2 - y^2 = a^2$

$$x\sec\theta - y\tan\theta = a \qquad (1)$$

Given lines are $x - y = 0$ (2)

and $x + y = 0$ (3)

Slove (1) and (2), (2) and (3), (3) and (1), we get vertices of the triangle as

$$\left(\frac{a}{\sec\theta - \tan\theta}, \frac{a}{\sec\theta - \tan\theta}\right),$$

$$\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-a}{\sec\theta + \tan\theta}\right) \text{ and } (0,0)$$

$\therefore$ Area of the triangle $= \frac{1}{2}\left|x_1y_2 - x_2y_1\right|$

$$= \frac{a^2}{2}\left(\frac{-1}{\sec^2\theta - \tan^2\theta} - \frac{1}{\sec^2\theta - \tan^2\theta}\right)$$

$$= \frac{a^2}{2}(-2) = a^2$$

15.Sol: Let the equation of line be $Y - y = M(X - x)$ given that it intersects the coordinate axis, that is

$X = x - \frac{y}{m}$ and $Y = y - mx$

Hence, $x - \dfrac{y}{m} = 2x \Rightarrow M = \dfrac{dy}{dx} = \dfrac{-y}{x}$

i.e., $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$

integrating both sides, we get

$$\int_{y} dy = -\int \frac{dx}{x}$$

i.e., $\ln y = -\ln x + c$

$\Rightarrow$ $\ln(yx) = c$

i.e., $xy = e^c$

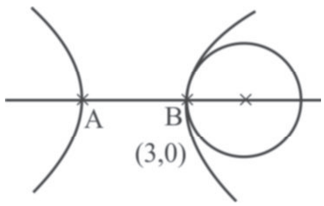
given that it is passing through (2, 4), that is $e^c = 8$

now $xy = 8$

This is rectangular hyperbola

hence the focus of $xy = 8$ are (4, 4) and $(-4,-4)$.

16.Sol:



$$16 = 9(e^2 - 1)$$

$$e = \frac{5}{3}$$

$$F_1 = (5,0)$$

Circle can be drawn touching hyperbola at point A or B only if Radius $= 2$

17.Sol: We have, for the given hyperbola $9 = 16(e^2 - 1) \Rightarrow e = \dfrac{5}{4}$ since (5, 0) satisfies the equation of the line $3x + (5 - 4\sqrt{2})y = 15,$ so the reflected ray must pass through $(-5,0)$ and $P = (4\sqrt{2},3)$

18.Sol: Given $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$.

The equation of tangent at point P is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

Slope of tangent $= \dfrac{b}{\tan\theta} \times \dfrac{\sec\theta}{a} = \dfrac{b}{a} \cdot \dfrac{1}{\sin\theta}$

Hence, the equation of perpendicular at P is

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

i.e., $by - b^2\tan\theta = -a\sin\theta\, x + a^2\tan\theta$

or $a\sin\theta x + by = (a^2 + b^2)\tan\theta$ (1)

Similarly the equation of perpendicular at Q is

$a\sin\phi x + by = (a^2 + b^2)\tan\phi$ (2)

On multiplying (1) by $\sin\phi$ and (2) by $\sin\theta$, we get

$$a\sin\theta\sin\phi x + b\sin\phi y = (a^2 + b^2)\tan\theta\sin\phi$$

$$a\sin\phi\sin\theta x + b\sin\theta y = (a^2 + b^2)\tan\phi\sin\theta$$

On subtraction we get by

$$(\sin\phi - \sin\theta) = (a^2 + b^2)(\tan\theta\sin\phi - \tan\phi\sin\theta)$$

$$\therefore \quad y = k = \frac{a^2 + b^2}{b} \cdot \frac{\tan\theta\sin\phi - \tan\phi\sin\theta}{\sin\phi - \sin\theta}$$

$$\because \quad \theta + \phi = \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{2} - \theta$$

$\Rightarrow \sin\phi = \cos\theta$ and $\tan\phi = \cot\theta$

$$\therefore \quad y = k = \frac{a^2 + b^2}{b} \cdot \frac{\tan\theta\cos\theta - \cos\theta\sin\theta}{\cos\theta - \sin\theta}$$

$$= \frac{a^2 + b^2}{b}\left(\frac{\sin\theta - \cos\theta}{\cos\theta - \sin\theta}\right) = -\frac{(a^2 + b^2)}{b}$$

19.Sol: Let $ax + by = 1$ be the chord

Making the equation of hyperbola homogeneous using (1), we get

$3x^2 - y^2 + (-2x + 4y)(ax + by) = 0$ (1)

$$(3 - 2a)x^2 + (-1 + 4b)y^2 + (-2b + 4a)xy = 0$$

Since the angle subtended at the origin is a right angle, so, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow (3 - 2a) + (-1 + 4b) = 0 \Rightarrow a = 2b + 1$

$\therefore$ The chords are $(2b + 1)x + by - 1 = 0$

or, $b(2 + y) + (x - 1) = 0,$

which, clearly, pass through the fixed point (1,-2).

Previous years JEE MAIN Questions

PERMUTATIONS & COMBINATIONS

[ONLINE QUESTIONS]

1. The number of ways in which 5 boys and 3 girls can be seated on a round table if a particular boy B_1 and a particular girl G_1 never sit adjacent to each other, is: **[2017]**
   (a) $5\times 6!$ (b) $6\times 6!$
   (c) 7! (d) $5\times 7!$

2. If all the words. with or without meaning, are written using the letters of the word QUEEN and are arranged as in English dictionary, then the position of the word QUEEN is : **[2017]**
   (a) 44^{th} (b) 45^{th} (c) 46^{th} (d) 47^{th}

3. The sum $\sum_{r=1}^{10}(r^2+1)\times(r!)$ is equal to: **[2016]**
   (a) $11\times(11!)$ (b) $10\times(11!)$
   (c) (11!) (d) $101\times(10!)$

4. If the four letter words (need not be meaningful) are to be formed using the letters from the word "MEDITERRANEAN" such that the first letter is R and the fourth letter is E. then the total number of all such words is : **[2016]**
   (a) 110 (b) 59 (c) $\frac{11!}{(2!)^3}$ (d) 56

5. If $\frac{^{n+2}C_6}{^{n-2}C_2}=11$, then n satisfies the equation: **[2016]**
   (a) $n^2+n-110=0$ (b) $n^2+2n-80=0$
   (c) $n^2+3n-108=0$ (d) $n^2+5n-84=0$

6. The value of $\sum_{r=1}^{15} r^2\left(\frac{^{15}C_r}{^{15}C_{r-1}}\right)$ is equal to : **[2016]**
   (a) 1240 (b) 560 (c) 1085 (d) 680

7. If in a regular polygon the number of diagonals is 54, then the number of sides of this polygon is **[2015]**
   (a) 12 (b) 6 (c) 10 (d) 9

8. The number of ways of selecting 15 teams from 15 men and 15 women. such that each team consists of a man and a woman, is: **[2015]**
   (a) 1120 (b) 1880 (c) 1960 (d) 1240

9. Two women and some men participated in a chess tournament in which every participant played two games with each of the other participants. If the number of games that the men played between themselves exceeds the number of games that the men played with the women by 66, then the number of men who participated in the tounament lies in the interval: **[2014]**

(a) [8,9] (b) [10,12) (c) (11,13] (d) (14,17)

10. 8-digit numbers are formed using the digits 1, 1, 2, 2, 2, 3, 4 and 4. The number of such number in which the odd digits do not occupy odd places is : **[2014]**

(a) 160 (b) 120 (c) 60 (d) 48

11. An eight digit number divisible by 9 is to be formed using digit from 0 to 9 without repeating the digits. The number of ways in which this can be done is : **[2014]**

(a) 72(7!) (b) 18(7!) (c) 40(7!) (d) 36(7!)

12. The sum of the digits in the unit's place of all the 4-digit numbers formed by using the numbers 3, 4, 5 and 6, without repetition, is : **[2014]**

(a) 432 (b) 108 (c) 36 (d) 18

13. 5- digit numbers are to be formed using 2, 3, 5, 7, 9 without repeating the digits. If p be the number of such numbers that exceed 20000 and q be the number of those that line between 30000 and 90000, then $p : q$ is : **[2013]**

(a) 6 : 5 (b) 3 : 2 (c) 4 : 3 (d) 5 : 3

14. On the sides AB, BC, CA of a ΔABC, 3, 4, 5 distinct points (excluding vertices A, B, C) are respectively chosen. The number of triangles that can be constructed using these chosen points as vertices are : **[2013]**

(a) 210 (b) 205 (c) 215 (d) 220

15. The number of ways in which an examiner can assign 30 marks to 8 questions, giving not less than 2 marks to any question, is : **[2013]**

(a) ${}^{30}C_1$ (b) ${}^{21}C_8$ (c) ${}^{21}C_7$ (d) ${}^{30}C_8$

16. A committee of 4 persons is to be formed from 2 ladies, 2 old men and 4 young men such that it includes at least 1 lady, at least 1 old man and at most 2 young men. Then the total number of ways in which this committee can be formed is: **[2013]**

(a) 40 (b) 41 (c) 16 (d) 32

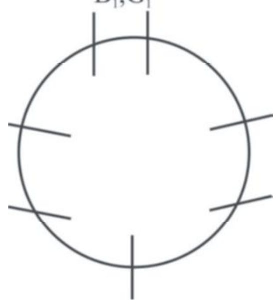
ANSWER KEY

1. a	2. c	3. b	4. b	5. c
6. d	7. a	8. d	9. b	10. b
11. d	12. b	13. d	14. b	15. c
16. b				

HINTS & SOLUTIONS

1.Sol: This problem can be easily done using complementary counting. Number of ways = Total - when B_1 and G_1 sit together

Total ways to seat 8 people on table $= 7!$



When B_1 and G_1 sit together $= 6! \times 2!$

Number of ways $= 7! - 2 \times 6! = 6!(7-2) = 5 \times 6!$

2.Sol: Alphabetic order of letters in the given word E, E, N, Q, U

(1) words starting with E : $4! = 24$

(2) words starting with N : $\frac{4!}{2} = 12$

(3) words starting with QE : $3! = 6$

(4) words starting with QN : $\frac{3!}{2!} = 3$

(5) words starting with $QUEEN$: $1! = 1$

therefore the required rank is

$24 + 12 + 6 + 3 + 1 = 46$

3.Sol: Given that $\sum_{r=1}^{10} (r^2+1)r!$

Let $T_r = (r^2+1)r!$

$$(r^2+1-r+r)r! = (r^2+r)r! - (r-1)!$$

$$= r(r+1)! - (r-1)r!$$

$$T_r = r(r+1)! - (r-1)r!$$

Now $T_1 = 1(2!) - 0$

$$T_2 = 2(3!) - 1(2!)$$
$$T_3 = 3(4!) - 2(3!)$$
$$\vdots = \vdots$$
$$T_{10} = 10(11!) - 9(10!)$$

which yields $\sum_{r=1}^{10}(r^2+1)r! = 10(11!)$

4.Sol: The first letter is R and the last one is E. Therefore, one has to find two more letters from the remaining 11 letters. Of the 11 letters, there are $2N's, 2E's$ and $2A's$ and one each of the remaining 5 letters.
The second and third positions can either have two different letters or have both the letters to be the same.
Case 1: When the two letters are different. One has to choose two different letters from the 8 available different choices. This can be done in $8\times7 = 56$ways.
Case 2: When the two letters are same. There are 3 options - the three can be either $N's$ or $E's$ or $A's$. Therefore, 3 ways.
Total number of possibilities $= 56 + 3 = 59$.

5.Sol: $\dfrac{n+2C_6}{n-2P_2} = 11$

$$\Rightarrow \frac{\dfrac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{6\cdot5\cdot4\cdot3\cdot2\cdot1}}{\dfrac{(n-2)(n-3)}{2\cdot1}} = 11$$

$$\Rightarrow (n+2)(n+1)n(n-1) = 11\cdot10\cdot9\cdot8$$

Now $\Rightarrow n = 9$

$$n^2 + 3n - 108$$
$$= (9)^2 + 3(9) - 108$$
$$= 81 + 27 - 108$$
$$= 108 - 108 = 0$$

6.Sol: $\sum_{r=1}^{15} r^2\left(\dfrac{^{15}C_r}{^{15}C_{r-1}}\right)$

$$\frac{^{15}C_r}{^{15}C_{r-1}} = \frac{\dfrac{15!}{(15-r)!r!}}{\dfrac{15!}{(16-r)!(r-1)!}} = \frac{(r-1)!(15-r)!(16-r)!}{(15-r)!r!}$$

$$= \frac{16-r}{r}$$

$$= \sum_{r=1}^{15} r^2\left(\frac{16-r}{r}\right) = \sum_{r=1}^{15} r(16-r)$$

$$= 16\sum_{r=1}^{15} r - \sum_{r=1}^{15} r^2$$

$$= \frac{16\times15\times16}{2} - \frac{15\times31\times16}{6}$$

$$= 8\times15\times16 - 5\times8\times31 = 1920 - 1240 = 680$$

7.Sol: Given that Number of diagonal $= 54$

We know that $\dfrac{n(n-3)}{2} = 54$

$$\Rightarrow n^2 - 3n - 108 = 0 \Rightarrow n^2 - 12n + 9n - 108 = 0$$
$$\Rightarrow n(n-12) + 9(n-12) = 0$$
$$\Rightarrow n = 12, -9 \Rightarrow n = 12 (\because n \neq -9)$$

8.Sol: The number of ways of choosing first couple is15 $\binom{15}{1}\cdot\binom{15}{1} = 15^2$,

Likewise second couple can be chosen in 1 $\binom{14}{1}\cdot\binom{14}{1} = 14^2$,

and similarly $3^{rd}, 4^{th}, ..., 15^{th}$ couples can be chosen in $13^2, 12^2, ...1^2$ ways respectively.
Thus the number of ways of choosing the couple is $15^2 + 14^2 + ... + 1^2 = \dfrac{15\times(15+1)(2\cdot15+1)}{6} = 1240$

9.Sol: Let there be n number of men and 2 women. Then the number of games that the men play between themselves is $2\cdot\binom{n}{2}$ and the number of games that the men played with the women is $2\cdot(2n)$

$\therefore \quad 2\cdot\binom{n}{2}-2\cdot 2n=66$

$\Rightarrow \quad n^2-5n-66=0$

i.e., $\quad n=11$

therefore the number of men participants is 11.

10.Sol: Odd digits cannot occupy odd places, so odd digits should occupy even places i,e 4 possiblities and odd numbers are three.

(1) So selecting three positions from 4 positions can be done in $\binom{4}{3}$ ways that is 4 ways.

(2) Filling the three positions with three numbers can be done is 3! ways but 1,1 are identical so total number of ways is $\frac{3!}{2!}$ i,e., 3

Now filling the remaining 5 positions can be done in 5! ways but again 2, 2, 2 are identical and 4, 4 are identical so number of ways is $\frac{5!}{3!\cdot 2!}$ i.e., in 10 ways so the total number of ways is $4\times 3\times 10=120$.

11.Sol: The sum of all the numbers between 0 and 9 is 45, hence a multiple of 9.

This means that if we want the sum of 8 numbers taken in this list to be multiple of 9, the sum of the two remaining numbers will also be a multiple of 9. The only possible pairs are (0, 9), (1, 8), (2, 7), (3, 6), (4, 5). when we select a given pair , we have then $8!=40320$ possibilities to form the required number with 8 digits

Since we have 5 different pairs, the answer is $5\cdot 8!$ If we remove leading 0 numbers , the number of possibilities become $7\cdot 7!$ for the last four pairs. Therefore the answer then becomes

$4\cdot 7\cdot 7!+8!=36\cdot 7!$

12.Sol: With each of the digits $\{3,4,5,6\}$ in the units place there are 3! four digits numbers are possible. Now the sum of the digits in the units place is

$$3!(3+4+5+6)=6\cdot 18=108$$

13.Sol: All the 5-digit numbers formed using these digits 2, 3, 5, 7 and 9 would be greater than 20000. So total number of such possible numbers $=5!$

And for numbers lying between 30000 and 90000 We have only 3 options to fill the first position i.e. ten thousands place and the rest 4 places can be filled in 4!, so total such ways $=3\cdot 4!$

Hence, $\quad p:q=5:3$

14.Sol: Any three points selected can not form a triangle if and only if all three are collinear".

Say 3, 4 and 5 points are marked on a triangle, altogether 12 points (excluding the three vertices of the original triangle)

Select any 3 points from all $12=\binom{12}{3}=220$

Number of collinear point selection cases

$=\binom{3}{3}+\binom{4}{3}+\binom{5}{3}$

where the three points have been selected fromthose points on the same side of the triangle. $1+4+10=15$ which yields, Total triangles =220-15=205.

15.Sol: First of all give 2 marks to each student (Total 16 marks). so now we have to distribute remaining 14 marks to 8 questions and all the question can get zero or more marks. So it will be same as distributing 14 coins among 8 beggars. So, Number of ways

$$\binom{14+8-1}{8-1}=\binom{21}{7}.$$

16.Sol: Since at least 1 lady, at least 1 old men and at most 2 young men must be chosen, we consider all committees which include 1, 1, 2, and 2 lady, with 1, 2, 1, and 2 old men, and 2, 1, 1, and 0 young men respectively. That is, we want to add the number of ways to:

• Choose 1 from 2 ladies, 1 from 2 old men, and 2 from 4 young men ${}^2C_1\times{}^2C_1\times{}^4C_2=21$

• Choose 1 from 2 ladies, 2 from 2 old men, and 1 from 4 young men ${}^2C_1\times{}^2C_2\times{}^4C_1=8$

• Choose 2 from 2 ladies, 1 from 2 old men, and 1 from 4 young men ${}^2C_2\times{}^2C_1\times{}^4C_1=8$

• Choose 2 from 2 ladies, 2 from 2 old men, and 0 from 4 young men ${}^2C_2\times{}^2C_2\times{}^4C_0=1$

Total $=24+8+8+1\;=41$

Synopticglance

COMPLEX NUMBERS

Introduction

Complex Numbers are a very popular and frequently used aspect of Mathematics. Composed of a real part and an imaginary part, they are written in the form $x+iy$. ***x*** denotes the real part and ***iy*** denotes the imaginary part. Complex numbers can be represented on an **Argand Diagram**. An **Argand Diagram** is similar to the **Cartesian Coordinate System** except that the Real axis and Imaginary axis replace the ***X and Y*** axis respectively which you would usually expect to see on the Cartesian system. This is shown in Figure 1.

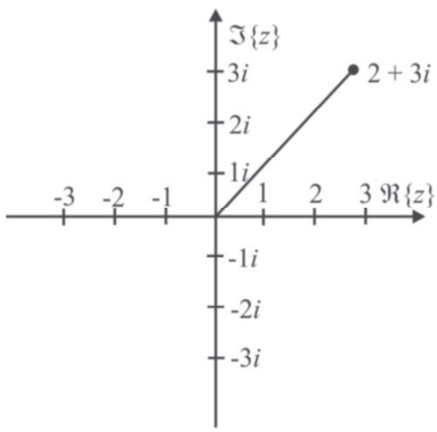


Fig-1: This Shows the Complex Number $2+3i$ plotted on an Argand diagram

(1) Imaginary Number

Square root of a negative real number is an imaginary number, while solving equation $x^2+1=0$ weget $x=\pm\sqrt{-1}$ which is imaginary, so the quantity $\sqrt{-1}$ is denoted by 'i' called 'iota' thus $i=\sqrt{-1}$.

e.g. $\sqrt{-2},\sqrt{-3},\sqrt{-4}$ .... may expressed as

$i\sqrt{2},\sqrt{3}i,2i$..........

(I) Properties of iota (i) :

$i=\sqrt{-1}$ *so* $i^2=-1, i^3=-i$ *and* $i^4=1$.

Hence, $n\in N, i^n=i,-1,-i,1$ attains four values according to the value of n, so

$i^{4n+1}=i,\quad i^{4n+2}=-1\quad i^{4n+3}=-i,\quad i^{4n+4}=1$

(II) Powers of the number i :

The formulas for the powers of a complex number with integer exponents are preserved for the algebraic form $z=x+iy$. Setting $z=i$, we obtain

$i^{4m}=1, i^{4m+1}=i, i^{4m+2}=-1, i^{4m+3}=-i, m\in Z$.

Hence, $i^m\in\{-1,1-i,i\}$ for all integers $m\ge 0$.

If m is a negative integer, we have

$$i^m=(i^{-1})^{-m}=\left(\frac{1}{i}\right)^{-m}=(-i)^{-m}.$$

(2) Basics

Definition: A complex number is a number of the format : $z=x+iy, x,y\in R$, where $i^2=-1$, x is a real part, iy is an imaginary part and y is coefficient of the imaginary part. The set of complex numbers includes the set of real numbers, and it is denoted by C.

Properties of Complex Numbers

- If $z=x+iy$, then the real part of z is denoted by Re (z) and the imaginary part by Im (z).
- A complex number is said to be purely imaginary if $Re(z)=0$.

- A complex number is said to be purely real if $\operatorname{Im}(z)=0$.
- The complex number $0=0+i0$ is both purely real and purely imaginary.
- Two complex numbers are said to be equal if and only if their real parts and imaginary parts are separately equal, i.e., $x_1+iy_1=x_2+iy_2$ implies $x_1=x_2$ and $y_1=y_2$.
- However, there is no order relation between complex and the expressions of the type $x_1+iy_1<(or>)x_2+iy_2$ are meaningless.

(I) The sum of the complex numbers

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then

$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$

(A) Properties concerning addition

The addition of complex numbers satisfies the following properties:

- Commutative law:

  $z_1+z_2=z_2+z_1$ for all $z_1,z_2\in C$
- Additive law:

  $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ for all $z_1,z_2,z_3\in C$
- Additive identity:

  There is a unique complex number $0=(0,0)$ such that $z+0=0+z$ for all $z=(x,y)\in C$.
- Additive inverse:

  For any complex number $z=(x,y)$ there is a unique $-z=(-x,-y)\in C$ such that $z+(-z)=(-z)+z=0$.

  The number $z_1-z_2=z_1+(-z_2)$ is called the difference of the z_1 and z_2. The operation that assigns to the numbers z_1 and z_2 the number z_1-z_2 is called subtraction and is defined by $z_1-z_2=(x_1+iy_1)-(x_2+iy_2)=(x_1-x_2)+i(y_1-y_2)\in C$.

(II) The multiplication of complex numbers

$z_1\cdot z_2=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$

(A) Properties concerning multiplication:

The multiplication of complex numbers satisfies the following properties.

- Commutative law

  $z_1\cdot z_2=z_2\cdot z_1$ for all $z_1,z_2\in C$
- Associative law

  $(z_1\cdot z_2)\cdot z_3=z_1\cdot(z_2\cdot z_3)$ for all $z_1,z_2,z_3\in C$.
- Multiplicative identity

  There is a unique complex number $1=(1,0)\in C$ such that $z\cdot 1=1\cdot z=z$ for all $z\in C$.
- Multiplicative inverse

  For any complex number $z=(x,y)\in C^*$ there is a unique number $z^{-1}=(x,y)\in C$ such that $z\cdot z^{-1}=1$
- Distributive law

  $z_1\cdot(z_2+z_3)=z_1\cdot z_2+z_1\cdot z_3$ for all $z_1,z_2,z_3\in C$.

(III) Quotient rule:

Two complex numbers $z_1=(x_1,y_1)\in C$ and $z=(x,y)\in C^*$ uniquely determine a third number called their quotient, denoted by $\frac{z_1}{z}$ and defined by

$$\frac{z_1}{z}=z_1\cdot z^{-1}=(x_1,y_1)\cdot\left(\frac{x}{x^2+y^2},-\frac{y}{x^2+y^2}\right)$$

$$=\left(\frac{x_1x+y_1y}{x^2+y^2},\frac{-x_1y+y_1x}{x^2+y^2}\right)\in C$$

The following properties hold for all complex numbers $z,z_1,z_2\in C^*$ and for all integer m,n.

- $z^m\cdot z^n=z^{m+n}$;
- $\frac{z^m}{z^n}=z^{m-n}$;
- $(z^m)^n=z^{mn}$;
- $(z_1\cdot z_2)^n=z_1^n\cdot z_2^n$;
- $\left(\frac{z_1}{z_2}\right)^n=\frac{z_1^n}{z_2^n}$.

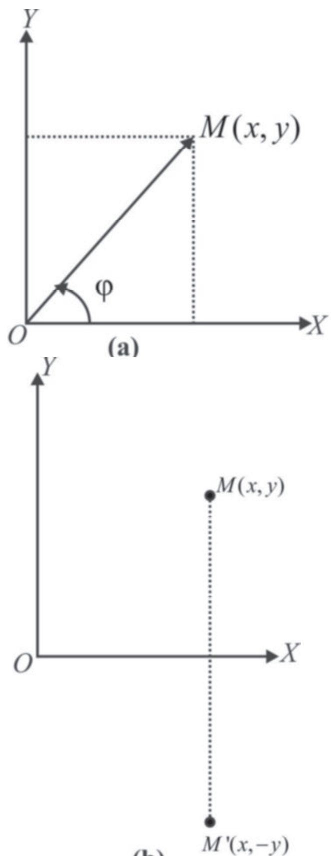
When $z=0$, we defined $0^n=0$ for all integers $n>0$.

(3) Geometrical Representation of Complex Numbers

There is bi-univocal correspondence between the set of complex numbers $z = x + iy$ and the set of complex numbers and the set of points $M(x, y)$ from plane.

Definition: The point $M(x, y)$ is called the geometric image of the complex number $z = x + yi$.

The complex number $z = x + yi$ is called the complex coordinate of the point $M(x, y)$. We will use the notation $M(z)$ to indicate that the complex coordinate of M is the complex number z.



(a)

(b)

Fig-2 (a) (b)

The geometric image of the complex conjugate $\bar{z}$ of complex $z = x + yi$ is the reflection point $M'(x, -y)$ across the x-axis of the point $M(x, y)$ (see the above Fig).

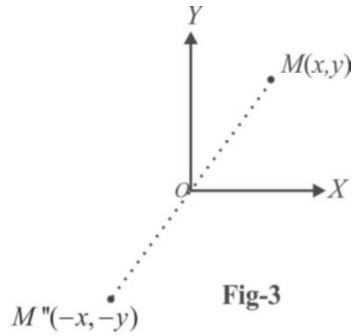


Fig-3

The geometric image of the additive inverse $-z$ of a complex number $z = x + yi$ is the reflection $M''(-x, -y)$ across the origin of the point $M(x, y)$.

The bijective function φ maps the set R onto the x - axis, which is called the real axis. On the other hand, the imaginary complex numbers correspond to the y-axis, which is called the imaginary axis. The plane $\prod$, whose points are identified with complex numbers, is called the complex plane.

On the other hand, we can also identify a complex number $z = x + yi$ with the vector $\vec{V} = \overline{OM}$, where $M(x, y)$ is the geometric image of the complex number z.

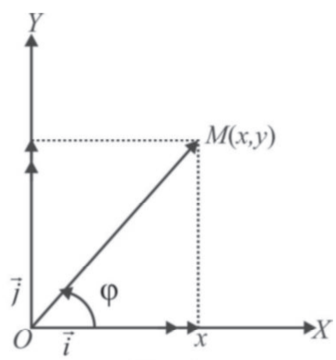


Fig: 4

Let V_0 be the set of vectors whose initial points are the origin O. Then we can define bijective function

$\varphi': C \to V_0, \varphi'(z) = \overrightarrow{OM} = \vec{v} = x\vec{i} + y\vec{j}$,

where $\vec{i}, \vec{j}$ are the vectors of the x-axis and y-axis, respectively.

(I) Geometric interpretation of the modulus:

Let us consider a complex number $z = x + yi$ and the geometric image $M(x, y)$ in the complex plane. The Euclidean distance OM is given by the formula

$$OM = \sqrt{(X_M - X_O)^2 + (Y_M - Y_O)^2}$$

hence, $OM = \sqrt{(x)^2 + (y)^2} = |z| = |\vec{v}|$. In other words, the absolute value $|z|$ of a complex number $z = x + yi$ is the length of the segment OM or the magnitude of the vector $\vec{v} = x + yi$.

Remarks :

- For a positive real number r, the set of complex numbers with moduli r corresponds in the complex plane to $C(O; r)$, our notation for the circle C with centre O and radius r.
- The complex numbers z with $|z| < r$ correspond to the interior points of circle C. On the other hand, the complex numbers z with $|z| > r$ correspond to the points in the exterior of circle C.

A complex plane is the plane in which we represent the complex numbers $z = x + iy$.

(II) Geometric Interpretation of the Algebraic Operations Addition and Substraction.

Consider the complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ and the corresponding vectors $\vec{v}_1 = x_1\vec{i} + y_1\vec{j}$ and $\vec{v}_2 = x_2\vec{i} + y_2\vec{j}$ observe that the sum of the complex numbers is

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i ,$$

Therefore, the sum $z_1 + z_2$ corresponds to the sum $\vec{v}_1 + \vec{v}_2$.

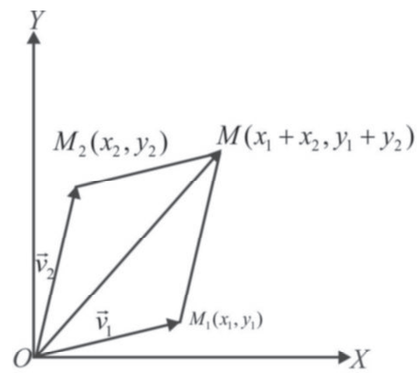


Fig-5

(III) Real multiples of a complex number:

Consider a complex number $z = x + iy$ and the corresponding vector $\vec{v} = x\vec{i} + y\vec{j}$. If λ is a real number, then the real multiple $\lambda z = \lambda x + i\lambda y$ corresponds to the vector

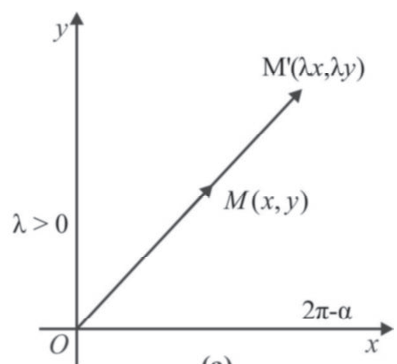
$\lambda\vec{v} = \lambda x\vec{i} + \lambda y\vec{j}$

Note that $\lambda > 0$ then the vectors $\lambda\vec{v}$ and $\vec{v}$ have the same orientation and $|\lambda\vec{v}| = |\lambda\vec{v}|$.

When $\lambda < 0$, the vector $\lambda\vec{v}$ changes to the opposite orientation and

$$|\lambda\vec{v}| = -\lambda|\vec{v}|.$$

Of course, if $\lambda = 0$, then $\lambda\vec{v} = \vec{0}$.



(a)

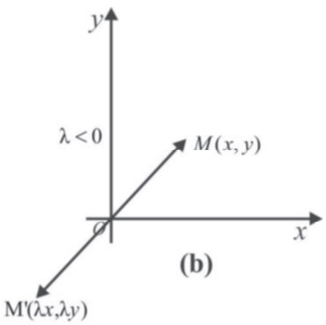


Fig-6 (a) (b)

(4) Trigonometric Representation

(I) Complex Numbers in Trigonometric form

A complex number $z = x + iy$ is represented by the point (x, y) in the complex plane. From the properties of complex numbers we can write

$$x = \mathrm{Re}(z) = |z|\cos(\varphi)$$

$$y = Im(z) = |z|\sin(\varphi),$$

where $|z| = \sqrt{x^2 + y^2}$. This is shown in the

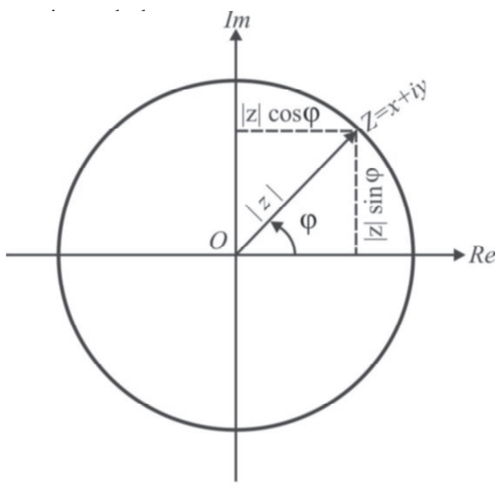


Fig-7

Note that the (x, y) pair can equivalently be described by trigonometric functions of another pair $(|z|, \varphi)$, often denoted by (r, φ). These are referred to as the polar coordinates of the complex number z. r is a non-negative number denoting the magnitude of the complex number (the radius of the circle) and is represented on the radial axis that extends outward from the origin at (0,0). An expression for φ can be obtained by dividing $y = |z|\sin(\varphi)$ by $x = |z|\cos(\varphi)$:

$$\varphi = \arctan\left(\frac{x}{y}\right)$$

and is called the argument of the complex number.

(II) Properties of Argument

- $\arg(z_1 z_2) = \varphi_1 + \varphi_2 = \arg(z_1) + \arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right) = \varphi_1 - \varphi_2 = \arg(z_1) - \arg(z_2)$
- $\arg(z^n) = n\arg(z), n \in I$

  In the above result $\varphi_1 + \varphi_2$ or $\varphi_1\varphi_2$ are not necessarily the principle values of the argument of corresponding complex numbers.
- $\arg(z) = 0, \pi, \Rightarrow z$ is a purely real number $\Rightarrow z = \bar{z}$
- $\arg(z) = \frac{\pi}{2}, \frac{-\pi}{2} \Rightarrow z$ is a purely imaginary number $\Rightarrow z = -\bar{z}$

(5) Exponential Representation

Knowledge POOL

◆ ***Polar and Rectangular Form***

Any complex number can be written in two ways, called rectangular form and polar form.

Rectangular : $z = x + iy$

Polar : $z = re^{i\varphi}$

To convert from one form to the other, use Euler's formula.

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

◆ ***To convert from polar to rectangular***

If you have a complex number in the form $re^{i\varphi}$ it is relatively straight forward to convert it into the form $x+iy$. Euler's formula says that $re^{i\varphi}$ is the same as $r\cos\varphi + ir\sin\varphi$. This is $x+iy$, where,

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

These are the same equations that we always convert a pair (r,φ) of polar coordinates to rectangular coordinates (x, y).

- ***To convert from rectangular to polar***

If we have a complex number in the form $x+iy$, we can convert it to the polar form $re^{i\varphi}$ by finding suitable values of r and φ. The radius r is easy to find using the formula

$$r = \sqrt{x^2+y^2}$$

The angle φ is a bit more difficult. We usually use the formula

$$\varphi = \arctan\frac{y}{x}$$

but there two limitations.

- If x is negative, add (or subtract) π to the result. This is because the arctan function always gives a value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so it will never give you ray pointing in the negative x direction.
- If $x=0$, then $\frac{y}{x}$ is not defined. In this case the point $x+iy$ lies on the y- axis (the imaginary axis). Then φ is either $\frac{\pi}{2}$ (if y is positive) or $-\frac{\pi}{2}$ (if y is negative).

These limitations are tricky at first, but they are not too hard to remember and try to picture where the point is (i.e., which quadrant).

- ***Geometric Interpretation of Euler's Formula***

Euler's formula allows for any complex number x to be represented as e^{ix}, which sits on a unit circle with real and imaginary components $\cos x$ and $\sin x$, respectively. Various operations (such as finding the roots of unity) can then be viewed as rotations along the unit circle.

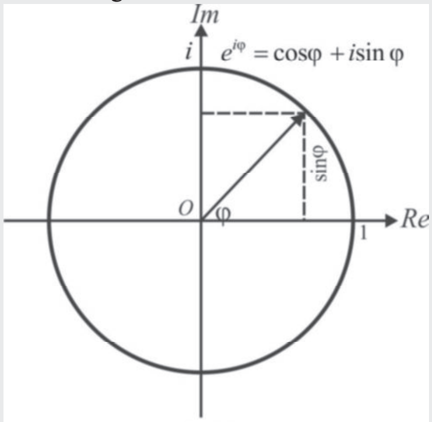


Fig-8

- ***Trigonometric Applications***

One immediate application of Euler's formula is to extend to definition of the trigonometric functions to allow for arguments that extend the range of the functions beyond what is allowed under the real numbers.

A couple of useful results to have at hand are the facts that

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi$$

so

$$e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi$$

It follows that

$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2}$$

and similarly

$$\sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}$$

and

$$\tan\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{i(e^{i\varphi}+e^{-i\varphi})}$$

(6) Important Theorems and Properties

(I) De Moivre's Theorem:

An important corollary of Euler's theorem is de Moivre's theorem.

Theorem (De Moivre's Theorem).

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$$

Applications:

De Moivre's theorem has many applications.

As an example, one may wish to compute the roots of unity, or the complex solution set to the equation $x^n = 1$ for integer n.

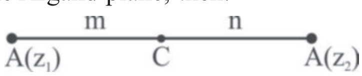
Notice that $e^{2\pi ki}$ is always equal to 1 for k an integer, so the n^{th} roots of unity must be

$$e^{2\pi ki/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

This process is initially difficult to divide the unit circle into n equally spaced wedges.

(II) Section Formula:

If points $A(z_1)$ and $B(z_2)$ represent the complex numbers z_1 and z_2 respectively in the Argand plane, then:



$C \equiv \left(\dfrac{mz_2 + nz_1}{m+n}\right)$ is the point dividing AB in the ratio $m : n$.

(III) $|z_1 \cdot z_2 \cdot z_3 .....z_n| = |z_1||z_2||z_3| \quad ....|z_n|$.

$\arg(z_1 \cdot z_2 \cdot z_3 .....z_n) = \arg(z_1) + \arg(z_2) + ... + \arg(z_n)$

When complex numbers are multiplied, their modulii get multiplied and their arguments get added together.

(IV) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \quad \arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

When two complex numbers are divided, their arguments are subtracted to get the argument of their quotient.

(V) **(i)** $\overline{z_1 + z_2 + z_3 + ... + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + ... + \overline{z_n}$

(ii) $\overline{z_1 \cdot z_2 \cdot z_3 .....z_n} = \overline{z_1} \cdot \overline{z_2} \cdot \overline{z_3} ....\overline{z_n}$

(iii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

(iv) $\left(\overline{z^n}\right) = \left(\overline{z}\right)^n$

(VI) $z + \overline{z} = 2\,\mathrm{Re}(z) \quad \Rightarrow \quad \overline{z} = -z$

if z is purely imaginary $(\because Re(z) = 0)$

$z - \overline{z} = 2i\,\mathrm{Im}(z) \quad \Rightarrow \quad \overline{z} = z$

if z is purely real $(\because Im(z) = 0)$

(VII) $z\overline{z} = |z|^2 \quad \Rightarrow \quad \overline{z} = \dfrac{1}{z}$ *if* $|z| = 1$

(VIII) $|-z| = |\overline{z}| = |z|$ and $\arg(\overline{z}) = -\arg(z)$

(IX) $|z^n| = |z|^n$

(X) $(z - z_0)$ is a factor of $f(z)$ *if and only if* $f(z_0) = 0$

- $z_1\overline{z_2} + \overline{z_1}z_2 = 2\,\mathrm{Re}(z_1\overline{z_2}) = 2\,\mathrm{Re}(\overline{z_1}z_2)$

  $\Rightarrow \quad z_1\overline{z_2} + \overline{z_1}z_2$ is purely real

- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z_2} + \overline{z_1}z_2)$

  $= |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(z_1\overline{z_2}) = |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(\overline{z_1}z_2)$

- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

- $-|z| \le \mathrm{Re}(z) \le |z|, -|z| \le \mathrm{Im}(z) \le |z|$

- Triangle Inequality :

  (i) $|z_1 + z_2| \le |z_1| + |z_2|$

  (ii) $|z_1 - z_2| \ge ||z_1| - |z_2||$

- $\dfrac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\tan\dfrac{\theta}{2}$

Finding the n^{th} roots of z

(1) The n^{th} Root of Unity

Let x be n^{th} root of unity. Then

$x^n = 1 = 1 + i0 = \cos 0^\circ + i\sin 0^\circ$

$= \cos(2k\pi + 0^\circ) + i\sin(2k\pi + 0^\circ)$

$= \cos 2k\pi + i\sin 2k\pi$ (where k is an integer)

$\Rightarrow x = \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n} k = 0, 1, 2, ...n-1$

Let $\omega = \cos 2\pi n + i\sin 2\pi n$. Then n^{th} root of unity are $\omega^t\,(t = 0, 1, 2, ..., n-1)$, i.e., the n^{th} roots of unity are $\omega, \omega^2, \omega^3, ...\omega^{n-1}$

(I) Properties of n roots of Unity Properties:

- Sum of n roots unity is zero

$$1+\omega+\omega^2+...+\omega^{n-1}=\frac{1-\omega^n}{1-\omega}=0$$

$$\Rightarrow \sum_{k=0}^{n-1}\cos\frac{2k\pi}{n}=0 \text{ and } \sum_{k=0}^{n-1}\sin\frac{2k\pi}{n}=0$$

Thus the sum of the roots of unity is zero.

- Sum of p^{th} power of n roots of unity is zero, if p is not a multiple of n

$$1+\omega^p+(\omega^2)^p+...+(\omega^{n-1})^p=\frac{1-(\omega^n)^p}{1-\omega^p}$$

$$=\frac{1-\left(e^{i\frac{2p\pi}{n}}\right)^n}{1-\omega^p}$$

$$=\frac{1-(e^{i-2p\pi})}{1-\omega^p}$$

- Sum of p^{th} power of n roots of unity is n, if p is a multiple of n

Let $p=\lambda n$, thus

$$\omega^p=e^{i\frac{2\pi p}{n}}=e^{i2\pi\lambda}=(\cos 2\pi\lambda+i\sin 2\pi\lambda)=1$$

$$1+\omega^p+(\omega^2)^p+...+(\omega^{n-1})^p=\frac{1-(\omega^n)^p}{1-\omega^p}$$

$$=1+1+1+...\,(n\ times)=n$$

- Product of the roots

$$1.\,\omega^p\cdot(\omega^2)^p\ldots(\omega^{n-1})^p=\omega^{\frac{n(n-1)}{2}}$$

$$=\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^{\frac{n(n-1)}{2}}$$

$$=\cos(n-1)\pi+i\sin(n-1)\pi$$

If n is even, $\omega^{\frac{n(n-1)}{2}}=-1$ and in case n is odd $\omega^{\frac{n(n-1)}{2}}=1$

- The points represented by the n n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having centre at the origin, one vertex being on the positive real axis (Geometrically represented as shown)

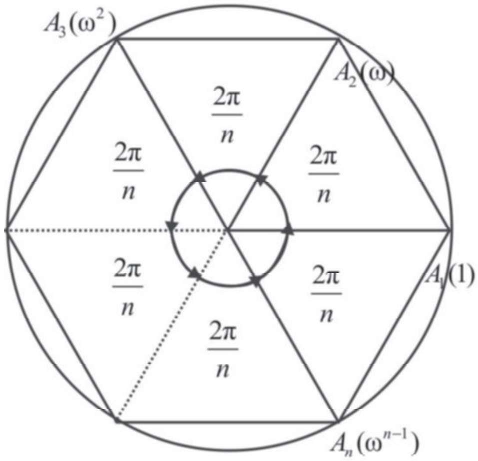


Fig-9

(II) Cube roots of unity:

For $n=3$, we get the cube roots of unity and they are $1, \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ and $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ i.e., $1, \frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$. They are generally denoted by $1, \omega$ and ω^2 and are geometrically represented by the vertices of an equilateral triangle whose circumcentre is the origin and circumradius is unity.

Properties of Cube Roots of Unity

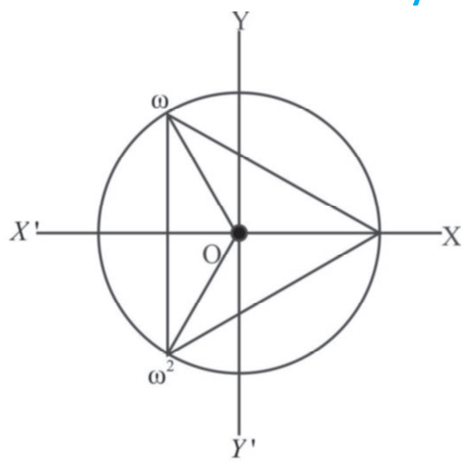


Fig-10

- $\omega^3 = 1$
- $1+\omega+\omega^2 = 0$
- $1+\omega^n+\omega^{2n} = 3$, where n is multiple of 3.
- $1+\omega^n+\omega^{2n} = 3$, (n is an integer, not a multiple of 3).
- $\omega = \dfrac{1}{\omega^2}$ and $\omega^2 = \dfrac{1}{\omega}$.
- $\omega = (\omega^2)^2$
- $\overline{\omega} = \omega^2$ and $\omega^2 = \overline{\omega}$

(2) Logarithm of Complex Number

In order to find $\log(x+iy)$, we write

$$\log(x+iy) = a+ib$$

$$\therefore x+iy = e^{a+ib} = e^a[\cos b + i\sin b]$$

$$= e^a(\cos(2k\pi+b)+i\sin(2k\pi+b))$$

$$\therefore e^a\cos(2k\pi+b) = x \text{ and}$$

$$e^a\sin(2k\pi+b) = y$$

Solve for a and b

$$\therefore e^{2a} = x^2+y^2 \quad \text{or } a = \frac{1}{2}\ln(x^2+y^2),$$

$$\tan(2k\pi+b) = \left(\frac{y}{x}\right)$$

When $k=0$, corresponding values of a and b are referred to as principle values.

(I) Method to Find

$(x+iy)^{a+ib}$ For evaluating $(x+iy)^{a+ib}$ we write

$$c+id = (x+iy)^{a+ib}$$

$$\therefore \log(c+id) = (a+ib)\cdot\log(x+iy)$$

Now evaluating $\log(x+iy)$ and then solve

$$c+id = e^{(a+ib)\log(x+iy)}$$

(3) Important Relations

- $x^2+y^2 = (x+yi)(x-yi)$
- $x^3+y^3 = (x+y)(x+\omega y)(x+\omega^2 y)$
- $x^3-y^3 = (x-y)(x-\omega y)(x-\omega^2 y)$
- $x^3+y^3+z^3-3xyz$

  $= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$
- $x^2+y^2+z^2-xy-yz-zx$

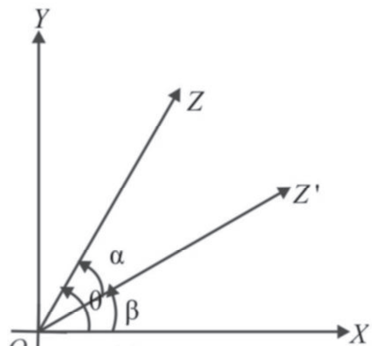
  $= (x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

Geometry via Complex Numbers

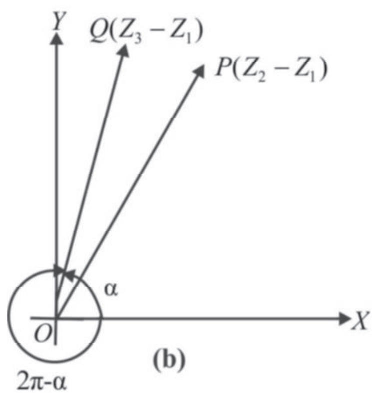
(1) Concept of Rotation

If z and z' are two complex numbers then argument of $\dfrac{z}{z'}$ is the angle through which oz must be turned in order that it may lie along oz.

$$\frac{z}{z'} = \frac{|z|e^{i\theta}}{|z'|e^{i\beta}} = \frac{|z|}{|z'|}e^{i\alpha}$$



(a)



(b)

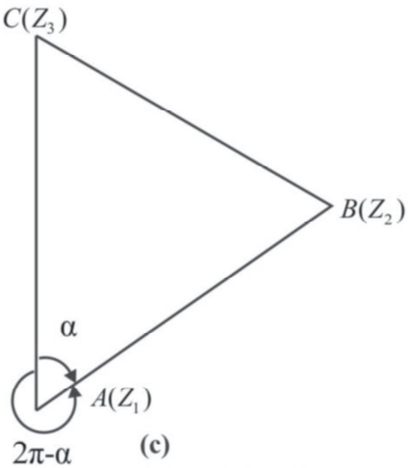


Fig-11 (a) (b) & (c)

In general, let z_1, z_2, z_3, be the three vertices of a triangle ABC described in the counter-clock wise sense. Drawn OP and OQ parallel and equal to AB and AC respectively. Then the point P is $z_2 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1}(\cos\alpha + i\sin\alpha) = \frac{CA}{BA}e^{i\alpha} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$$

Note that $\arg(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$ is the angle through which OP must be rotated in the anti- clockwise direction so that it becomes parallel to OQ.

Here we can write $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$

In this case we are rotating OP in clockwise direction by an angle $(2\pi - \alpha)$. Since rotation is in clockwise direction, we are taking negative sign with angle $(2\pi - \alpha)$.

(2) Section formula

Let z_1 and z_2 be any two complex numbers representing the points A and B respectively in the argand plane. Let C be the point dividing the line segment AB internally in the ratio $m : n$, i.e., $\dfrac{AC}{BC} = \dfrac{m}{n}$ and let the complex number associated with C be z. Let us rotate the line BC about the point C so that it becomes parallel to CA. The corresponding equation of rotation will be

$$\frac{z_1 - z}{z_2 - z} = \frac{|z_1 - z|}{|z_2 - z|}e^{i\pi} = \frac{m}{n}(-1)$$

$$\Rightarrow nz_1 - nz = -mz_2 + mz$$

$$\Rightarrow z = \frac{nz_1 + mz_2}{m + n}$$

Similarly if $C(z)$ divides the segment AB externally in the ratiio of $m : n$

$$\Rightarrow z = \frac{nz_1 - mz_2}{m - n}$$

In the specific case, if $C(z)$ is the mid-point of AB then $z = \dfrac{z_1 + z_2}{2}$

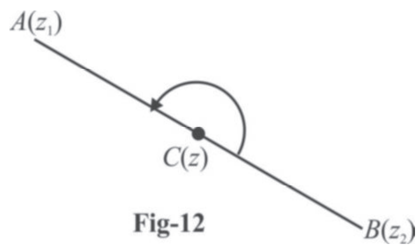


Fig-12

(3) Condition for Collinearity

If there are three real numbers (other than 0) l, m and n such that $lz_1 + mz_2 + nz_3 = 0$ and $l + m + n = 0$ then complex numbers z_1, z_2 and z_3 will be collinear.

(4) Equation of a straight line

Writing $x = \dfrac{z + \bar{z}}{2}, y = \dfrac{z - \bar{z}}{2}$ etc. and rearranging terms, we find that the equation of the line through z_1 and z_2 is given by

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \text{ or } \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

(5) Equation of a straight line with help of rotation formula

Let $A(z_1)$ and $B(z_2)$ be any two points lying on any line and we have to obtain the equation of this line. For this purpose let us take any point $C(z)$ lying on this line. Since

$$\arg\frac{z-z_1}{z_2-z_1}=0 \text{ or } \pi .$$

$$\frac{z-z_1}{z_2-z_1}=\frac{\overline{z}-\overline{z_1}}{\overline{z_2}-\overline{z_1}}$$

This is the equation of the line that passes through $A(z_1)$ and $B(z_2)$. After rearranging the terms, it can also be put in the following form

$$\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix}=0$$

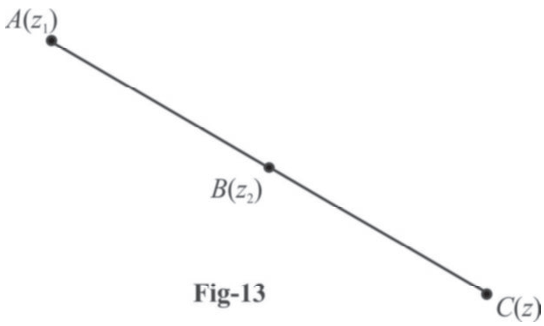


Fig-13

(I) General equation of the line:

From equation (1) we get,

$$z\left(\overline{z_2}-\overline{z_1}\right)-z_1\overline{z_2}+z_1\overline{z_1}=\overline{z}(z_2-z_1)-\overline{z_1}z_2+\overline{z_1}z_1$$

$$\Rightarrow z\left(\overline{z_2}-\overline{z_1}\right)+\overline{z}(z_1-z_2)+\overline{z_1}z_2-z_1\overline{z_2}=0$$

Here, $\overline{z_1}z_2-z_1\overline{z_2}$ is a purely imaginary number as $\overline{z_1}z_2-z_1\overline{z_2}=2i\,\text{Im}(\overline{z_1}z_2)$.

Let $\overline{z_1}z_2-z_1\overline{z_2}=ib, b\in\mathbb{R}$

$$\Rightarrow z(\overline{z_2}-\overline{z_1})+\overline{z}(z_1-z_2)+ib=0$$

$$\Rightarrow zi(\overline{z_2}-\overline{z_1})+\overline{z}i(z_2-z_1)-b=0$$

Let $a=i(z_2-z_1)$

$$\Rightarrow \overline{a}=i(\overline{z_1}-\overline{z_2})$$

$$\Rightarrow z\overline{a}+\overline{z}a+b=0$$

This is the general equation of a line in the complex plane.

(II) Slope of a given line:

Let the given line $\Rightarrow z\overline{a}+\overline{z}a+b=0$.

Replacing z by $x+iy$, we get

$$(x+iy)\overline{a}+(x-iy)a+b=0$$

$$\Rightarrow x(\overline{a}+a)+iy(\overline{a}-a)+b=0$$

Its slope is $\dfrac{\overline{a}+a}{i(\overline{a}-a)}=\dfrac{2\,\text{Re}(a)}{2i^2\,\text{Im}(a)}=-\dfrac{\text{Re}(a)}{\text{Im}(a)}$

(III) Equation of a line parallel to given line:

Equation of a line parallel to the line $z\overline{a}+\overline{z}a+b=0$ is $z\overline{a}+\overline{z}a+\lambda=0$ (where λ is a real number)

(IV) Equation of a line perpendicular to given line:

Equation of a line perpendicular to the line $z\overline{a}+\overline{z}a+b=0$ is $z\overline{a}-\overline{z}a+i\lambda=0$ (where λ is a real number)

(V) Equation of Perpendicular Bisector:

Consider a line segment joining $A(z_1)$ *and* $B(z_2)$. Let the line 'L' be its perpendicular bisector. If $P(z)$ be any point on the 'L', we have

$$PA=PB\Rightarrow|z-z_1|=|z-z_2|$$

$$\Rightarrow|z-z_1|^2=|z-z_2|^2$$

$$\Rightarrow(z-z_1)(\overline{z}-\overline{z}_1)=(z-z_2)(\overline{z}-\overline{z}_2)$$

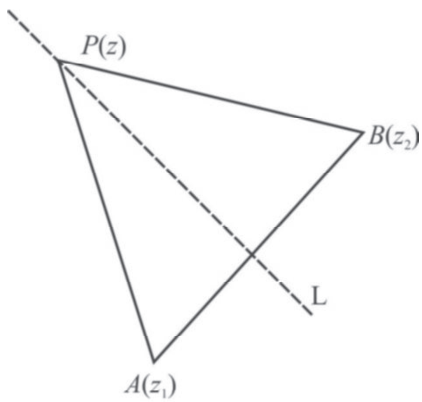


Fig-14

$$\Rightarrow z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1$$

$$= z\bar{z} - z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$$

$$\Rightarrow z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + z_1\bar{z}_1 - z_2\bar{z}_2 = 0$$

(VI) Perpendicular Distance of a given point from a given line

Let the given line be $z\bar{a} + z\bar{a} + b = 0$ and the given point z_c. Saying $z = x_c + iy_c$.

Replacing z by $x + iy$, in the given equation, we get,

$$x(a + \bar{a}) + iy(\bar{a} - a) + b = 0$$

Distance of (x_c, y_c) from this line is

$$\frac{|x_c(a + \bar{a}) + iy_c(\bar{a} - a) + b|}{\sqrt{(a + \bar{a})^2 - (a - \bar{a})^2}}$$

$$= \frac{|z_c\bar{a} + \bar{z}_c a + b|}{\sqrt{4(\mathrm{Re}(a))^2 + 4(im(a))^2}}$$

$$= \frac{|z_c\bar{a} + \bar{z}_c a + b|}{2|a|}$$

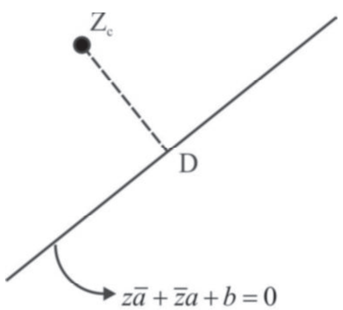


Fig-15

Note:

$\arg(z - z_0) = \theta$ represents a line passing through z_0 with slope $\tan\theta$. (making angle θ with the positive direction of x - axis)

(6) Equation of a Circle

Consider a fixed complex number z_0 and let z be any complex number which moves in such a way that it's distance from z_0 is always equals to 'r'. This implies z would lie on a circle whose centre is z_0 and radius r. And its equation would be $|z - z_0| = r$.

$$\Rightarrow |z - z_0|^2 = r^2$$

$$\Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

$$\Rightarrow z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 - r^2 = 0$$

Let $-a = z_0$ and $z_0\bar{z}_0 - r^2 = b$

$$\Rightarrow z\bar{z} + a\bar{z} + a\bar{z} + b = 0$$

It represents the general equation of a circle in the complex plane.

(I) Properties of Circles

- $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represent a circle whose centre is $-a$ and radius is $\sqrt{a\bar{a} - b}$.

  Thus $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, $(b \in R)$ represents a real circle if and only if $a\bar{a} - b \geq 0$

- Now let us consider a circle described on a line segment $AB, (A(z_1), B(z_2))$ as diameter. Let $P(z)$ be any point on the circle. As the angle in the semicircle is $\pi/2, \angle APB = \pi/2$.

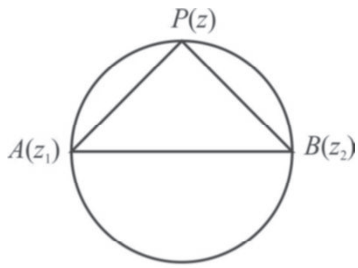


Fig-16

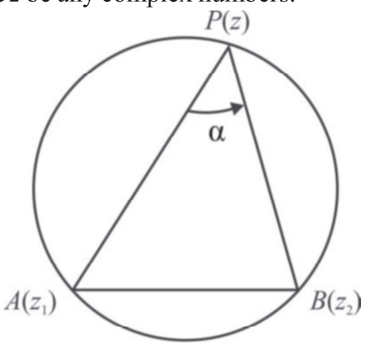
$$\Rightarrow \arg\left(\frac{z_1 - z}{z_2 - z}\right) = \pm\pi/2$$

$$\Rightarrow \frac{z - z_1}{z - z_2} \text{ is purely imaginary}$$

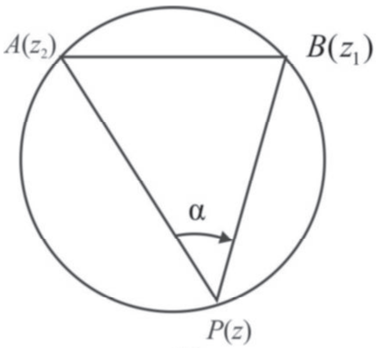
$$\Rightarrow \frac{z - z_1}{z - z_2} + \frac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2} = 0$$

$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$

- Let z_1 and z_2 be two given complex numbers and z be any complex numbers.



(a)



(b)

Fig-17 (a) (b)

Such that $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\alpha$, where $\alpha \in (0,\pi)$

Then 'z' would lie on an arc of segment of a circle on z_1z_2, containing angle α. Clearly if $\alpha \in (\pi/2), z$ would lie on the major arc (excluding the points z_1 and z_2) and $\alpha \in (\pi/2,\pi),'z'$ would lie on the minor arc (excluding the points z_1 and z_2.

Note:

The sign of α determines the side of z_1z_2 on which the segment lies. Thus α is positive in fig 17(a) and negative in fig 17(b).

- Let ABCD be a cyclic quadrilateral such that $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ lie on a circle. Clearly $\angle A+\angle C=\pi$.

$$\Rightarrow \arg\left(\frac{z_4-z_1}{z_2-z_1}\right)+\arg\left(\frac{z_2-z_3}{z_4-z_3}\right)=\pi$$

$$\Rightarrow \arg\left(\frac{z_4-z_1}{z_2-z_1}\right)\left(\frac{z_2-z_3}{z_4-z_3}\right)=\pi$$

$$\Rightarrow \frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)} \text{ is purely real.}$$

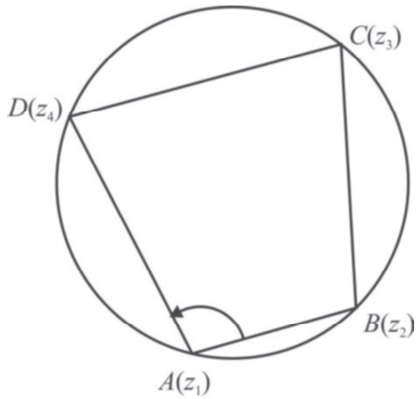


Fig-18

(II) Equation of Tangent To A Given Circle

Let $|z-z_0|=r$ be the given circle and we have to obtain the tangent at $A(z_1)$. Let us take any point $P(z)$ on the tangent line at $A(z_1)$.

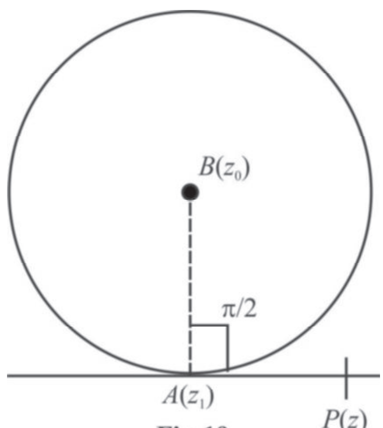


Fig-19

Clearly $\angle PAB=\pi/2$

$$\arg\left(\frac{z-z_1}{z_0-z_1}\right)=\pm\frac{\pi}{2}$$

$\Rightarrow \dfrac{z-z_1}{z_0-z_1}$ is purely imaginary

$$\Rightarrow \frac{z-z_1}{z_0-z_1}+\frac{\bar{z}-\bar{z}_1}{\bar{z}_0-\bar{z}_1}=0$$

$$\Rightarrow (z-z_1)(\bar{z}_0-\bar{z}_1)+(\bar{z}-\bar{z}_1)(z_0-z_1)=0$$

$$\Rightarrow z(\bar{z}_0-\bar{z}_1)+\bar{z}(z_0-z_1)+z_1\bar{z}_1-z_1\bar{z}_0+z_1\bar{z}_1 -\bar{z}_1 z_0=0$$

$$\Rightarrow z(\bar{z}_0-\bar{z})+\bar{z}(z_0-z_1)+2|z_1|^2 -z_1\bar{z}_0-\bar{z}_1z_0=0$$

In particular if given circle is $|z|=r$, equation of the tangent at $z=z_1$ would be,

$$z\bar{z}_1+\bar{z}z_1=2|z_1|^2=2r^2$$

If $\left|\dfrac{z-z_1}{z-z_2}\right|=\lambda$ $(\lambda\in R^+,\lambda\neq 1)$,

where z_1 and z_2 are the given complex numbers and z is an arbitary complex number then z would lie on a circle.

Note :

- If we take 'C' to be the mid-point of A_2A_1, it can be easily proved that $CA\cdot CB=(CA_1)^2$ i.e., $|z_1-z_0||z_2-z_0|=r^2$. where the point C is denoted by z_0 and r is the radius of the circle.
- If $\lambda=1 \Rightarrow |z-z_1|=|z-z_2|$ hence $P(z)$ would lie on the right bisectors of the line $A(z_1)$ and $B(z_2)$. Note that in this case z_1 and z_2 are the mirror images of each other with respect to the right bisector.

IMPORTANT POINTS

Some important results to remember

- The triangle whose vertices are the points represented by complex numbers z_1, z_2, z_3 is equilateral if $\dfrac{1}{z_2-z_3}+\dfrac{1}{z_3-z_1}+\dfrac{1}{z_1-z_2}=0$, i.e., if $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$.
- $|z-z_1|+|z-z_2|=\lambda$, represents an ellipse if $|z_1-z_2|<\lambda$, having the points z_1 and z_2 as its foci. And if $|z-z_2|=\lambda$, then z lies on a line segment connecting z_1 and z_2.
- $||z-z_1|-|z-z_2||=\lambda$, represents a hyperbola if $|z_1-z_2|>\lambda$, having the points z_1 and z_2 as its foci. And if $|z-z_2|=\lambda$, z lies on the passing z_1 and z_2 excluding the points between z_1 and z_2.

MULTIPLE CHOICE QUESTIONS

1. Let n be a positive integer. Then $(i)^{4n+1}+(-i)^{4n+5}=$
   (a) 0 (b) $2i$ (c) i (d) $-i$
2. The complex number $\dfrac{1+2i}{1-i}$ lies in the quadrant number
   (a) I (b) II (c) III (d) IV
3. The imaginary part of i^i is
   (a) 0 (b) 1 (c) 2 (d) -1
4. If z_1 and z_2 are complex numbers satisfying $\left|\dfrac{z_1+z_2}{z_1-z_2}\right|=1$ and $\arg\left(\dfrac{z_1-z_2}{z_1+z_2}\right)=\pm n\pi$ $(n\in\mathbb{Z})$ then $\dfrac{z_1}{z_2}$ is always
   (a) Zero
   (b) A rational number
   (c) A positive real number
   (d) A purely imaginary number
5. If Z_1, Z_2 are two complex numbers satisfying $\left|\dfrac{Z_1-3Z_2}{3-Z_1Z_2}\right|=1, |z_1|\neq 3$ then $|z_2|=$

(a) 1 (b) 2 (c) 8 (d) 4

6. If $\left|z-\frac{4}{z}\right|=2$, then the maximum value of $|z|$ is equal to

(a) $\sqrt{3}+1$ (b) $\sqrt{5}+1$

(c) 2 (d) $2+\sqrt{2}$

7. For all complex numbers z_1, z_2 satisfying $|z_1|=12$ and $|z_2-3-4i|=5$, find the minimum value of $|z_1-z_2|$

(a) 2 (b) 25 (c) 22 (d) 10

8. If z_1 and z_2 are purely real then $z_1, z_2, \bar{z}_1, \bar{z}_2$ form

(a) Parallelogram (b) Square

(c) Rhombus (d) Straight line

9. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)$ equals

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

10. The number of solutions of the equation $z^2+\bar{z}=0$ is

(a) 1 (b) 2 (c) 3 (d) 4

11. If $cis\,\alpha$ is a solution of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+....+p_n=0$ then the value of $p_1\sin\alpha+p_2\sin 2\alpha+...+p_n\sin n\alpha=$

(a) 0 (b) n (c) $2n$ (d) n^2

12. If $z^2+z+1=0$, where z is a complex number then the value of $\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+...+\left(z^6+\frac{1}{z^6}\right)^2$ is

(a) 18 (b) 54 (c) 6 (d) 12

13. The common values of 6^{th} roots of unity and cube roots of unity are

(a) $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ (b) $1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$

(c) $1, \frac{-1+3i}{2}, \frac{-1-3i}{2}$ (d) $1, \frac{1+3i}{2}, \frac{1-3i}{2}$

14. If α is a non-real roots of $x^6=1$ then $\frac{\alpha^5+\alpha^3+\alpha+1}{\alpha^2+1}=$

(a) α^2 (b) 0 (c) $-\alpha^2$ (d) α

15. Let $f(x)=x^4+x^3+x^2+x+1$ find the remainder when $f(x^5)$ is divided by $f(x)$.

(a) 0 (b) 1 (c) 2 (d) 5

16. If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are distinct fifth unity evaluate two expression below

$$\frac{31}{2-\alpha_1}+\frac{31}{2-\alpha_2}+\frac{31}{2-\alpha_3}+\frac{31}{2-\alpha_4}$$

(a) 49 (b) $4a$

(c) 31 (d) None of these

17. There are 24 different complex numbers z such that $z^{24}=1$. For how many of these is z^6 a real number?

(a) 0 (b) 4 (c) 6 (d) 12

18. If $|z+2+3i|=5$ then the locus of z is

(a) A circle with centre (2,3) and radius 25 units

(b) A circle with centre (-2,-3) and radius 25 units

(c) A circle with centre (2,3) and radius 5 units

(d) A circle with centre (-2,-3) and radius 5 units

19. The area of the triangle with vertices $z, iz, z+iz$ is 50 then $|z|=$

(a) 0 (b) 5 (c) 10 (d) 15

20. Reflection of the line $\bar{a}z+a\bar{z}=0$ in the real axis is

(a) $\overline{az}+az=0$ (b) $\frac{\bar{a}}{a}+\frac{\bar{z}}{z}=0$

(c) $(a+\bar{a})(z+\bar{z})=0$ (d) $az=0$

21. In the Argand diagram P denotes Z.

If $\left|\frac{z-4i}{z-2}\right|=2$ then the locus of P is

(a) The perpendicular bisector of P_1, P_2 where $P_1=4i$ and $P_2=2$

(b) A line perpendicular to P_1P_2 cutting it in the ratio 2:1

(c) A circle

(d) A line parallel to P_1P_2

22. Let there be an equilateral triangle on the complex plane with vertices $z_1, z_2 z_3$. Let the circumcentre of the triangle be z_0. If $z_0 \neq 0$, find the value of

$$\frac{z_1^2+z_2^2+z_3^2}{z_0^2}$$

(a) 0 (b) 1 (c) 2 (d) 3

23. If z_1, z_2 are the roots of $z^2+az+b=0$ and z_1, z_2, origin be the vertices of an equilateral triangle then $a^2-3b=$

(a) 0 (b) 1 (c) -1 (d) 2

24. The series $\sum_{n=0}^{\infty}\frac{\cos n\theta}{2^n}$, when $\theta=\frac{\pi}{3}$, converges to

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) None of these

25. If $|z|<\sqrt{2}-1$, then $|z^2+2z\cos\alpha|$, where α is real is

(a) Less than 1 (b) $\sqrt{2}+1$

(c) $\sqrt{2}-1$ (d) None of these

26. The complex number $Z=\begin{vmatrix} 2 & 3+i & -3 \\ 3-i & 0 & -1+i \\ -3 & -1-i & 4 \end{vmatrix}$ is

(a) $3-4i$ (b) $5+4i$

(c) $-5i$ (d) A real number

27. If α is a root of $z^5+z^3+z+3=0$, then

(a) $|\alpha|\geq 1$ (b) $|\alpha|<1$

(c) α lies completely outside the unit circle

(d) α lies inside the unit circle $|z|=1$

28. If $|z-2|=\min\{|z-1|,|z-5|\}$ where z is a complex number then

(a) $\text{Re}(z)=\frac{3}{2}$ (b) $\text{Re}(z)=\frac{7}{2}$

(c) $\text{Re}(z)\in\left\{\frac{3}{2},\frac{7}{2}\right\}$ (d) None of these

29. If $|a_k|<3, 1\leq k\leq n$, then all the complex numbers z satisfying the equation

it $a_1z+a_2z^2+....+a_nz^n=0$

(a) Lie out side the circle $|z|=\frac{1}{4}$

(b) Lie inside the circle $|z|=\frac{1}{4}$

(c) Lie on the circle $|z|=\frac{1}{4}$

(d) Lie in $\frac{1}{3}<|z|=\frac{1}{2}$

ANSWER KEY

1. a	**2.** b	**3.** a	**4.** d	**5.** a
6. b	**7.** a	**8.** d	**9.** a	**10.** c
11. a	**12.** d	**13.** a	**14.** c	**15.** d
16. a	**17.** d	**18.** d	**19.** c	**20.** a
21. c	**22.** d	**23.** a	**24.** c	**25.** a
26. d	**27.** a	**28.** b	**29.** a	

HINTS & SOLUTIONS

1.Sol: We have, $i^1=i, i^2=-1, i^3=-i$ and $i^4=1$,

i.e., $i^{4n+1}=i$ and $(-i)^{4n+5}=-i$

Now, $(i)^{4n+1}+(-i)i-i=0$

2.Sol: Given that $\frac{1+2i}{1-i}$

$$\Rightarrow \frac{(1+2i)(1+i)}{1^2-i^2}=\frac{1}{2}(3i-1)$$

$$\therefore\ a+ib=-\frac{1}{2}+\frac{3}{2}i$$

Now, $\theta=\tan^{-1}\left(\frac{b}{a}\right)$

$=\tan^{-1}(-1)$

$\therefore$ Since a is negative, therefore θ lies in second quadrant.

3.Sol: Let $A=i^i$

$\Rightarrow\ \log A = i\log i$

i.e., $\Rightarrow\ \log A = i\log i$

$$\Rightarrow \quad \log A = i \log e^{i\frac{\pi}{2}}$$

$$\Rightarrow \quad \log A = i\, i\frac{\pi}{2} = -\frac{\pi}{2}$$

$$\therefore \quad A = e^{-\frac{\pi}{2}}$$

4.Sol: Given that $\left|\dfrac{z_1+z_2}{z_1-z_2}\right| = 1$

$$\Rightarrow \quad \left|\frac{\frac{z_1}{z_2}+1}{\frac{z_1}{z_2}-1}\right| = 1$$

i.e., $\dfrac{\frac{z_1}{z_2}+1}{\frac{z_1}{z_2}-1} = \cos\alpha + i\sin\alpha$

$$\Rightarrow \quad \frac{2\frac{z_1}{z_2}}{2} = \frac{1+\cos\alpha + i\sin\alpha}{\cos\alpha + i\sin\alpha - 1}$$

i.e., $\dfrac{z_1}{z_2} = -i\cot\left(\dfrac{\alpha}{2}\right)(\alpha \pm n\pi)$

$\therefore$ $\dfrac{z_1}{z_2}$ is purely imaginary.

5.Sol: $|z_1 - 3z_2| = |3 - z_1\overline{z_2}|$

$$\Rightarrow (z_1 - 3z_2)(\overline{z_1} - 3\overline{z_2}) = (3 - z_1\overline{z_2})(3 - \overline{z_1}z_2)$$

6.Sol: Given $\left|z - \dfrac{4}{z}\right| = 2$

we have $\left|z - \dfrac{4}{z}\right| \geq |z| - \left|-\dfrac{4}{z}\right|$

i.e., $\quad 2 \geq |z| - \dfrac{4}{|z|}$

$$\Rightarrow \quad |z|^2 - 2|z| - 4 \leq 0$$

$$\therefore \quad |z| \leq \frac{2 \pm \sqrt{20}}{2}$$

$$|z| \leq 1 \pm \sqrt{5}$$

That is $|z| \leq \sqrt{5} + 1$.

7.Sol: We have $|z_1 - z_2| \geq |z_1| - |z_2|$

$|z_2 - (3+4i)| \geq |z_2| - 5$

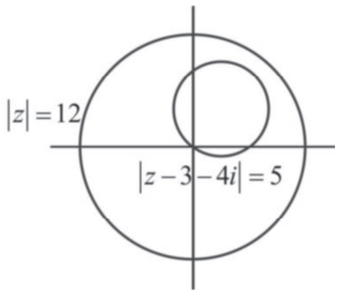
i.e., $|z_2| \leq |z_2 - (3+4i)| + 5$

$|z_2| \leq 10$

Now, $|z_1 - z_2| \geq |z_1| - |z_2|$

$\geq 12 - 10$

$|z_1 - z_2| \geq 2$



8.Sol: Given that z_1, z_2 are purely real i.e., $\overline{z_1}, \overline{z_2}$ also real, z_1, z_2, $\overline{z_1}, \overline{z_2}$ are on real line.

9.Sol: Given that z_1 and z_2 are conjugate

i.e., $z_2 = \overline{z_1}$

Also, z_3 and z_4 are conjugate complex numbers.

i.e., $z_4 = \overline{z_3}$

Now, $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right) = \arg\left(\dfrac{z_1}{z_4}\right)\left(\dfrac{z_2}{z_3}\right)$

$$= \arg\left(\frac{z_1}{\overline{\overline{z_3}}}\right)\left(\frac{\overline{z_1}}{z_3}\right)$$

$$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right) = 0$$

10.Sol: Method-1

Let $z = re^{i\theta}$. Then the equation becomes

$r^2 e^{2i\theta} + r = 0$

so either $r = 0$ or $re^{2i\theta} = -1$

$re^{2i\theta}=-1 \quad \Rightarrow \quad r=1$ and $2\theta=\pi+2n\pi$

putting r,θ values in z.

we get $z=0$ or $z=i,-i$

Method-2

Given that $z^2+|z|=0$

$$\Rightarrow \quad z^2=-|z|$$

$$\Rightarrow \quad |z|^2=|-|z||=|z|$$

so, $|z|^2-|z| \quad \Rightarrow \quad |z|(|z|-1)=0$

$$\Rightarrow \quad |z|=0,|z|=1$$

If $|z|=0$. Then put into above equation

$$z^2+0=0 \quad \Rightarrow z=0$$

If $|z|=1$, Then put into above equation

$$z^2+1=0 \quad \Rightarrow \quad z=\pm i.$$

11.Sol: Given that $Cis\alpha$ is a solution of the equation

$$x^n+p_1^{x^{n-1}}+p_2^{x^{n-2}}+.....+p_n=0$$

$$\Rightarrow \quad 1+p_1\left(\frac{1}{x}\right)+p_2\left(\frac{1}{x^2}\right)+.....+p_n\frac{1}{x^n}=0$$

put $x=Cis\alpha$

$$1+p_1Cis(-\alpha)+p_2Cis(-2\alpha)+.....+p_nCis(-n\alpha)=0$$

We get,

$$p_1\sin\alpha+p_2\sin 2\alpha+.....+p_n\sin n\alpha=0$$

12.Sol: Given that, $z^2+z+1=0$

$$\Rightarrow \quad z+\frac{1}{z}=-1$$

$$\Rightarrow \quad z^2+\frac{1}{z^2}=-1$$

Similarly $z^3+\frac{1}{z^3}=2, z^4+\frac{1}{z^4}=-1$, $z^5+\frac{1}{z^5}=-1$

and

$$z^6+\frac{1}{z^6}=2$$

Now, $\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+.....+\left(z^6+\frac{1}{z^6}\right)^2$

$$=1+1+4+1+1+4$$

$$=12$$

13.Sol: $x^3=1 \Rightarrow x=1, x^2+x+1=0$

$$\Rightarrow x=1,\frac{-1\pm\sqrt{3}i}{2}$$

$$x^6=1 \Rightarrow x^6-1=0$$

$$\Rightarrow (x^3-1)(x^3+1)=0$$

$\therefore$ common roots are $+1$,

$$\frac{-1+\sqrt{3}i}{2},\frac{-1-\sqrt{3}i}{2}$$

14.Sol: Given that α, is a non real roots of $x^6=1$

i.e., $(x-1)(x^5+x^4+x^3+x^2+x+1)=0$

$$\Rightarrow \quad \alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha+1=0$$

i.e., $\alpha^5+\alpha^3+\alpha+1=-\alpha^2(\alpha^2+1)$

$$\frac{\alpha^5+\alpha^3+\alpha+1}{\alpha^2+1}=-\alpha^2$$

15.Sol: Given that $f(x)=x^4+x^3+x^2+x+1$

Therefore $f(x^5)=x^{20}+x^{15}+x^{10}+x^5+1$

we have $(1-x)(x^4+x^3+x^2+x+1)=1-x^5$

$$\Rightarrow \quad x^5=1+(x-1)\left(x^4+x^3+x^2+x+1\right)$$

$$=1+(x-1)f(x)$$

It should be noted that $x^{5n}=1\bmod f(x)$.

Therefore, the remainder of $\dfrac{x^{20}+x^{15}+x^{10}+x^5+1}{f(x)}$

is 5.

16.Sol: If $1,\alpha_1,\alpha_2,....,\alpha_n$ are the roots of $f(x)=0$,

than $\dfrac{f'(x)}{f(x)}=\sum_{i=1}^{n}\dfrac{1}{x-a_i}$

consider $f(x)=x^5-1$ here and using the above property, we have

$$\frac{5x^4}{x^5-1}=\frac{1}{x-1}+\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\frac{1}{x-\alpha_3}+\frac{1}{x-\alpha_4}$$

putting $x=2$, we get

$$\frac{80}{31}=\frac{1}{1}+\frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\frac{1}{2-\alpha_3}+\frac{1}{2-\alpha_4}$$

$$\Rightarrow \frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\frac{1}{2-\alpha_3}+\frac{1}{2-\alpha_4}=\frac{80-31}{31}$$

i.e., $\frac{31}{2-\alpha_1}+\frac{31}{2-\alpha_2}+\frac{31}{2-\alpha_3}+\frac{31}{2-\alpha_4}=49$

17.Sol: Note that these z such that $z^{24}=1$ are $e^{\frac{ni\pi}{2}}$

for integer $0\le n<24$. So $z^6=e^{\frac{ni\pi}{2}}$.

This is real if $\frac{n}{2}\leftarrow z\Leftrightarrow$ (n is even)

thus the answer is the number of even $0\le n<24$ which is 12

Method-2

from the fundamental theorem of algebra that $z^{24}=1$ must have 24 solutions or notice that the question is simply referring to the 24^{th} roots of unity of which we know there must be 24. Notice that $1=z^{24}=(2^6)^4$, so for any solution z, z^6 will be one of the 4^{th} roots of unity $(1, i, -1, \text{or } -i)$.

Then 6 solutions of z will satisfy $z^6=1, 6$ will satisfy $z^6=-1$, so there must be 12 such z.

18.Sol: Let $z=x+iy$

Given $|z+2+3i|=5$

$\Rightarrow |(x+2)+i(y+3)|=5$

$\Rightarrow (x+2)^2+(y+3)^2=5^2$

19.Sol: Let $z=z+iy, iz=-y+ix$

and $z+iz=(x-y)+i(x+y)$

Then the area of the triangle formed by the vertices $(x,y),(-y,x),(x-y,x+y)$ is

$$\frac{1}{2}\begin{vmatrix} x-y & x-y & x \\ y & x & x+y & y\end{vmatrix}=50$$

$$\frac{1}{2}\left|x^2+y^2-xy-y^2-x^2+xy+xy-y^2-x^2-xy\right|=50$$

$$\frac{1}{2}\left|-x^2-y^2\right|=50$$

$\Rightarrow \left|x^2+y^2\right|=100$

i.e., $|z|=10$

20.Sol: Let $a=\alpha+i\beta$ and $z=x+iy$, then $\bar{a}z+a\bar{z}=0$ becomes $\alpha x+\beta y=0$ or $y=\left(\frac{-\alpha}{\beta}\right)x$.

Its reflection in the x-axis is $y=\frac{\alpha}{\beta}x$ or $\alpha x-\beta y=0$

$\Rightarrow \left(\frac{a+\bar{a}}{2}\right)\left(\frac{z+\bar{z}}{2}\right)-\left(\frac{a-\bar{a}}{2i}\right)\left(\frac{z-\bar{z}}{2i}\right)=0$

$\Rightarrow az+\bar{a}\,\bar{z}=0$

21.Sol: Let $z=x+iy$

Given that $\left|\frac{z-4i}{z-2}\right|=2$

i.e., $\left|\frac{x+(y-4)i}{(x-2)+yi}\right|=2$

$\Rightarrow x^2+(y-4)^2=4(x-2)^2+y^2$

$\Rightarrow 3x^2+3y^2-16x+8y=0$

22.Sol: The circumcentre of an equilateral triangle is

$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

So, $\frac{z_1^2+z_2^2+z_3^2}{z_0^2}$

$$=\frac{z_1z_2+z_2z_3+z_3z_1}{z_1^2+z_2^2+z_3^2+2(z_1z_2+z_2z_3+z_3z_4)}=3$$

23.Sol: Given that z_1, z_2 are the roots of $z^2+az+b=0$ By using vieta's theorem, we get i.e., $z_1+z_2=-a$ and $z_1z_2=b$ also given that $0, z_1, z_2$ are vertices of an equilateral triangle

i.e., $z_1^2+z_2^2=z_1z_2$

$\Rightarrow (z_1+z_2)^2=3z_1z_2$

i.e., $a^2-3b=0$

24.Sol: We have $\left(1-\frac{e^{i\theta}}{2}\right)^{-1}=1+\frac{e^{i\theta}}{2}+\frac{e^{i2\theta}}{2^2}+...$

$$\Rightarrow \frac{2}{2-\cos\theta - i\sin\theta} = 1 + \frac{e^{i\theta}}{2} + \frac{e^{i2\theta}}{2^2} + \ldots$$

$$\frac{2(2-\cos\theta + i\sin\theta)}{5+4\cos\theta} = 1 + \frac{e^{i\theta}}{2} + \frac{e^{i2\theta}}{2^2} + \ldots$$

Now equating the real parts we get

$$\frac{2(2-\cos\theta)}{5-4\cos\theta} = \sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$$

put $\theta = \frac{\pi}{3^2}$

$$\Rightarrow \frac{2\left(2-\frac{1}{2}\right)}{5-4\times\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n} = 1$$

25.Sol: Given that $|z| < \sqrt{2} - 1$

Now, $|z^2 + 2z\cos\alpha| \le |z^2| + |2z\cos\alpha|$

$\le |z|^2 + 2|z|$

$< 2 + 1 - 2\sqrt{2} + 2\sqrt{2} - 2$

< 1

$\therefore \quad |z^2 + 2z\cos\alpha| < 1$

26.Sol:

$$\bar{Z} = \begin{vmatrix} 2 & 3-i & -3 \\ 3+i & 0 & -1-i \\ -3 & -1+i & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3+i & -3 \\ 3-i & 0 & -1+i \\ -3 & -1-i & 4 \end{vmatrix} = Z$$

$\therefore$ Z is real.

27.Sol: We have $\alpha^5 + \alpha^3 + \alpha + 3 = 0$

i.e., $\alpha^5 + \alpha^3 + \alpha = -3$

Suppose $|\alpha| < 1$

$|-3| = |\alpha^5 + \alpha^3 + \alpha| \le |\alpha|^5 + |\alpha|^3 + |\alpha| < 1+1+1 = 3$

which is a contradiction $\quad \therefore |\alpha| \ge 1$

$\Rightarrow$ α lies either on or outside the unit circle.

28.Sol: The given condition is

$|z-2| = \min\{|z-1|, |z-5|\}$ there arises two cases:

Case-I

If $\min\{|z-1|, |z-5|\} = |z-1|$

then $|z-2| = |z-1|$

$\Rightarrow \quad |z-2|^2 = |z-1|^2$

i.e., $(z-2)(\bar{z}-2) = (z-1)(\bar{z}-1)$

$z\bar{z} - 2(z+\bar{z}) + 4 = z\bar{z} - (z+\bar{z}) + 1$

i.e., $z + \bar{z} = 3$

$\therefore \quad 2\,\text{Re}(z) = 3$

$\Rightarrow \quad \text{Re}(z) = \frac{3}{2}$

Case-II

$\min\{|z-1|, |z-5|\} = |z-5|$

then $|z-2| = |z-5|$

$\Rightarrow \quad |z-2|^2 = |z-5|^2$

$\Rightarrow \quad (z-2)(\bar{z}-2) = (z-5)(\bar{z}-5)$

$\Rightarrow \quad 3(z+\bar{z}) = 21$

$3 \times 2\,\text{Re}(z) = 21$

$\text{Re}(z) = \frac{21}{6} = \frac{7}{2}$

29.Sol: Given that $|a_k| < 3, 1 \le k \le n$

$\Rightarrow \quad |a_1| < 3, |a_2| < 3, \ldots\ldots |a_n| < 3$

Now, $1 + a_2 z + a_2 z^2 + \ldots. + a_n z^n = 0$

$\Rightarrow \quad |a_1 z + a_2 z^2 + \ldots a_n z^n| = |-1|$

i.e., $1 = a_1 z + a_2 z^2 + \ldots + a_n z^n$

$|a_1||z| + |a_2||z|^2 + \ldots + |a_n||z|^n$

i.e., $1 < 3\left(|z| + |z|^2 + \ldots. |z|^n\right) < 3\left(|z| + |z|^2 + \ldots.\infty\right)$

$\Rightarrow \quad 1 < \frac{3|z|}{1-|z|}$

$\Rightarrow \quad 1 - |z| < 3|z|$

i.e., $|z| > \frac{1}{4}$

OLYMPIAD Primer

This article aims at solving entry level Math Olympiad (Pre-RMO in india). We have compiled some of the most useful results and tricks in geometry that helps in solving problems at this level.

Basic Techniques:

Knowing the basic facts and important therorems well is important for solving geometry problems, but still insufficient. In fact, it is common to see beginners who diligently learn many theorems, but do not know how to apply those results and slove geometry problems. Indeed, many beginners are not aware of the commonly used *technique* (instead of theorems), which are not found in most textbooks.

The following is an elementary example: **NO** advanced knowledge is required to solve this problem. Can you see the clues without referring to the solution?

1. Given a quadrilateral $ABCD$ where $AD = BC$ and $\angle BAC + \angle ACD = 180°$, show that $\angle B = \angle D$.

1.Sol: If $\angle BAC = \angle ACD = 90°$, we have $\Delta BAC \cong \Delta DCA$ and hence, $ABCD$ is a parallelogram and $\angle B = \angle D$.

Suppose $\angle BAC < 90°$. Let DC extended and AB extended intersect at E. Since $\angle BAC + \angle ACD = 180°$, we have $\angle BAC = \angle ECA$ and $AE = CE$. Choose F on the line CD such that $AF = AD$. We have $\angle D = \angle AFD$. Now $BC = AD = AF$ gives $\Delta ABC \cong \Delta CFA$ (S.A.S.). It follows that $\angle B = \angle AFD = \angle D$.

If $\angle BAC > 90°$, the lines AB and CD intersect at the other side of AC and a similar argument applies.

Basic and commonly used techniques in solving geometry problems include the following:

Cut and Paste

When given equal line segments, equal or supplementary angles, and sum of angles or line segments which are far apart, one may cut and paste, moving those angles or line segments together. This technique may give staright lines, isosceles triangles or congruent triangles.

Construct congruent and similar triangles

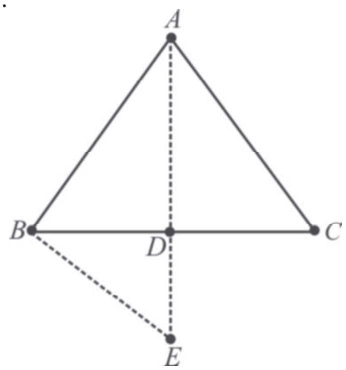
One strategy to show equal angles or line segments is to place them in congruent or similar triangles. If no such triangles exist in the diagram, consider drawing auxiliary lines and construct one! Notice that any other angles or line segments known to be equal may give inspiration on which triangles could be congruent or similar.

Reflection about an angle bisector

When given an angle bisector, it is naturally a line of symmetry. Reflecting about the angle bisector may bring angles and line segments together and hence, it may be an effective technique besides "cut and paste".

Double the median

Refer to the diagram below. Given ΔABC and its median AD, extending AD to E with $AD = DE$ gives ΔABE where $BE = AC$ and $\angle ABC = 180° - \angle A$.

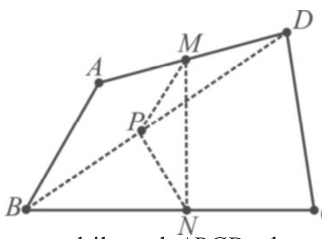


Hence, $\sin\angle A = \sin\angle ABE$ and $[\Delta ABC] = [\Delta ABE]$.

Moreover, (twice) the median of ΔABC becomes a side of ΔABE. This may be useful technique when constructing congruent and similar traingles.

Midpoints and Midpoint Theorem

When midpoints are given, it is natural to apply the Midpoint Theorem, which not only gives parallel lines, but also moves the (halved) line segments around. In particular, if connecting the midpoints does not give a midline of the triangle, one may choose more midpoints and draw the midlines. Refer to the diagram below.



Given a quadrilateral $ABCD$ where M, N are the midpoints of AD, BC, respectively, simply connecting MN does not give any conclusion. If we choose P, the midpoint of BD, then

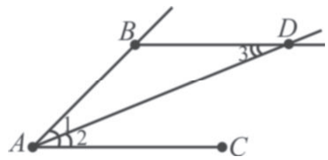
$$PM = \frac{1}{2}AB \text{ and } PN = \frac{1}{2}CD.$$

If we know more about AB and CD, say $AB = CD$, then we conclude that ΔPMN is an isosceles triangle.

On the other hand, if midpoints are given together with right angled triangle, one may consider the median on the hypotenuse.

Angle bisector plus parallel lines

One may easily see an isosceles triangle from an angle bisector plus parallel lines. Refer to the diagram below. If AD bisects $\angle A$, we have $\angle 1 = \angle 2$.



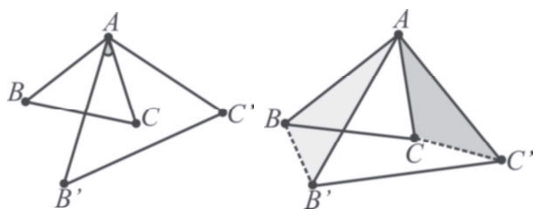
If $AC // BD$, $\angle 2 = \angle 3$. It follows that $AB = CD$. Notice that this technique could also be applied reversely. In the diagram above, if we know $AB = BD$, then by showing $AC // BD$, we conclude that AD bisects $\angle A$.

Similar triangles sharing a common vertex

A pair of similar triangles sharing a common vertex may immediately give another pair of similar triangles. Refer to the following diagrams where $\Delta ABC \sim \Delta AB'C'$.

Since $\frac{AB}{AB'} = \frac{AC}{AC'}$ and $\angle BAC = \angle B'AC'$, by sbtracting $\angle B'AC$, we see that $\angle BAB' = \angle CAC'$. It follows that $\Delta ABB' \sim \Delta ACC'$.

Notice that this technique applies for the inverse as well. If we have $\Delta ABB' \sim \Delta ACC'$, we may also conclude that $\Delta ABC \sim \Delta AB'C'$.



Angle-chasing

This is an elementary but effective technique when we explore angles related to a circle, especially when an incircle or circumcircle of a triangle is given (because the incenter and circumcenter give us even more equal angles). If more than one circle is given, it is a basic technique to apply the angle properties repeatedly and identify equal angles far apart or apparently unrelated. Indeed, experienced contestants are very familiar with the angle properties and are sharp in observing and catching equal angles.

However, one should avoid long-winded angle-chasing which leads nowwhere. If that happens, one may seek clues from the line segments instead, say identifying similar triangles, or applying the intersecting Chords Theorem and the Tangent Theorem.

Watch out for right angles

When right angles are given, it is worthwhile to spend time and effort digging out more information about them, because right angles may lead to a number of approaches.

(1) If a right angled triangle with a height on the hypotenuse is given, we will have similar triangles.
(2) If there are other heights or the orthocenter of a triangle, we may find parallel lines.
(3) One may see concyclicity when a few right angles are given.
(4) If a right angle is extended on the circumference of a circle, it corresponds to a diameter of the circle.

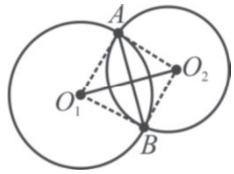
One should always refer to the context of the problem and determine which approach might be effective.

Perpendicular bisector of achord

Introducing a perpendicular from the center of a circle to a chord is a simple technique but occasionally, it may be decisively useful. Notice that the perpendicular bisector gives both right angles and the midpoint of the chord.

Draw a line connecting the centers of two intersecting circles

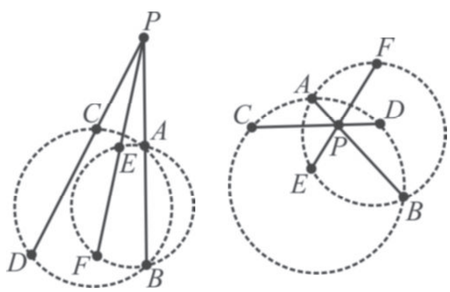
This is a very basic technique where the line connecting the centers of the two circles is a line of symmetry.



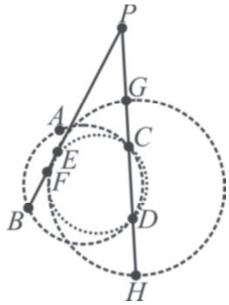
Refer to the diagram above. Notice that $O_1O_2 \perp AB$ and O_1O_2 is the angle bisector of both $\angle AO_1B$ and $\angle AO_2B$. Even though this is an elementary result, one may apply it to solve difficult problems.
It is noteworthy that beginners tend to overlook this elemenatary property during problem solving, especially when the diagram is complicated.

Relay: Tangent Secant Theorem and Intersecting Chords Theorem

When more than one circle is given and there is a common chord or concurrency, one may apply the Tangent Secant Theorem or the Intersecting Chords Theorem repeatedly to acquire more concyclicity. Refer to the diagrams below. Can you see *C, D, E, F* are concyclic in both diagrams? can you see that $PC \cdot PD = PA \cdot PB = PE \cdot PF$?



Refer to the diagram below. If *A*, *B*, *C*, *D* are concyclic, *C, D, E, F* are concyclic and *E, F, G, H* are concyclic, can you see that *A, B, G, H* are concyclic? (**Hint**: $PA \cdot PB = PC \cdot PD = PE \cdot PF = PG \cdot PH$.)



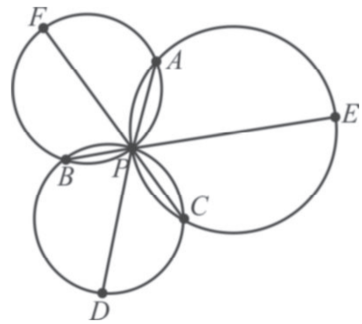
We shall illustrate these techniques with more examples in this section.

Exercise

1. Given a quadrilateral *ABCD*, the external angle bisectors of $\angle CAD, \angle CBD$ intersect at *P*. Show that if $AD + AC = BC + BD$, then $\angle APD = \angle BPC$.
2. Given an acute angled triangle $\Delta ABC, AD$ is the angle bisector of $\angle A, BE$ is a median and *CF* is a height. Show that *AD*, *BE*, *CF* are concurrent if and only if F lies on the perpendicular bisector of *AD*.
3. Given a quadrilateral *ABCD* inscribed inside $\odot O$, draw lines l_1, l_2 such that l_1 and the line *AB* is symmetric about the angle bisector of $\angle CAD$, and l_2 and the line *AB* is symmetric about the angle bisector of $\angle CBD$. If l_1 and l_2 intersect at *M*, show that $OM \perp CD$.
4. Let *I* be the incenter of ΔABC. *M*, *N* are the midpoints of *AB*, *AC* respectively. *NM* extended and *CI* extended intersect at *P*. Draw $QP \perp MN$ at P such that *QN // BI*. Show that $QI \perp AC$.
5. Let *O, G* denote the circumcenter and the centroid of ΔABC respectively. Let the perpendicular bisectors of *AG*, *BG*, *CG* intersect mutually at *D*, *E*, *F* respectively. Show that *O* is the centroid of ΔDEF.
6. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three circles such that Γ_1, Γ_2 intersect at *A* and P, Γ_2, Γ_3 intersect at C and P, and Γ_1, Γ_3 intersect at *B* and *P*.

   Refer to the following diagram. If *AP* extended intersects Γ_3 at *D, BP* extended intersects Γ_2 at *E* and *CP* extended intersects Γ_1 at *F*, show that

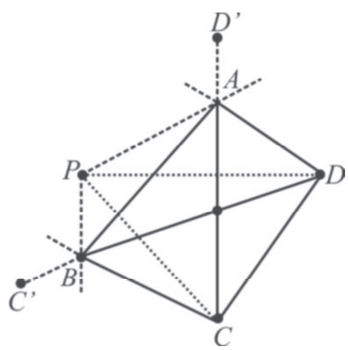
   $$\frac{AP}{AD} + \frac{BP}{BE} + \frac{CP}{CF} = 1.$$

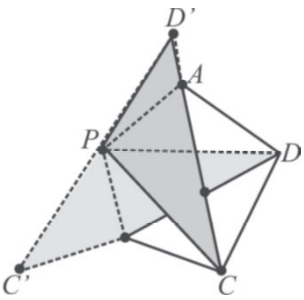


7. Let *AB* be the diameter of a semicircle centered at *O*. Given two points *C, D* on the semicircle, *BP* is tangent to the circle, intersecting *CD* extended at *P*. If the line *PO* intersects *CA* extended and *AD* extended at *E, F* respectively, show that *OE* = *OF*.
8. Given an acute angled triangle ΔABC where *AD*, *BE, CF* are heights, draw $FP \perp DE$ at *P*. let *Q* be the point on *DE* such that *QA* = *QB*. Show that $\angle PAQ = \angle PBQ = \angle PFC$.

HINTS & SOLUTIONS

1.Sol:





Extend DB to C' such that $BC' = BC$ and extend CA to D' such that $AD = AD'$. Can you see that C, C' are symmetric about the line PB, and D, D' are symmetric about the line PA? (Hint: $\Delta BCC'$ is an isosceles triangle and PB is the perpendicular bisector of CC') Now $BC = BC'$ and $AD = AD'$. Refer to the right digram above. Can you see that $AD + AC = BC + BD$ implies $CD' = C'D$? Can you see that $PC = PC'$, $PD = PD'$ and hence, $\Delta PC'D \cong \Delta PCD'$?

Now $\angle C'PD = \angle CPD'$ and the conclusion follows.

2.Sol: By Ceva's Theorem, AD, BE, CF are concurrent if and only if $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$. Since BE *is a* median, it is equivalent to $\frac{AF}{BF}=\frac{CD}{BD}$, or $DF//AC$.

We claim that $DF \,//\, AC$ if and only $AF = DF$. In fact, since AD bisects $\angle A$, $DF//AC$ if and only if $\angle ADF = \angle CAD = \angle BAD$, which is equivalent to $AF = DF$.

In conclusion, AD, BE, CF are concurrent if and only if $AF = DF$, i.e., F lies on the perpendicular bisecot of AD.

3.Sol: Let P be the midpoint of $\overparen{CD}$. Clearly, AP, BP are the angle bisectors of $\angle CAD, \angle CBD$ respectively.

Let l_1 and l_2 intersect $\odot O$ at A, E and B, F respectively. Since l_1 and AB are symmetric about AP, we must have $\angle BAP = 180° - \angle EAP = \angle ECP$ (because A,E,C,P are concyclic). (1)

Since P is the midpoint of $\overparen{CD}$, we have $\angle PCD = \angle PAC$. (2)

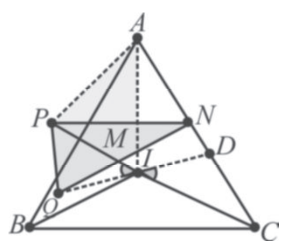
(1) and (2) imply that $\angle BAP - \angle PAC = \angle ECP - \angle PCD$, which gives $\angle BAC = \angle DCE$, i.e, $\overparen{BC}$ and $\overparen{DE}$ extend the same angle on the circumference. This implies $BC = DE$ and hence, $BCDE$ is an isosceles trapezium with $BE//CD$.

Since l_2 and AB are symmetric about BP, a similar argument applies which gives $AF \,//\, CD$ and $ADCF$ is an isosceles trapezium. Now it is easy to see that $AEBF$ is also an isosceles trapezium. Notice that $AM = MF$ and hence, OM is the perpendicular bisector of AF. Since $AF \,//\, CD$. we must have $OM \perp CD$.

4.Sol: Since CI bisects $\angle C$ and $BC//MN$, we have $\angle NCP = \angle BCP = \angle NPC$, i.e., $PN = CN$. Since N is the midpoint of AC, we have $PN = AN = CN$. and hence, $\angle APC = 90°$

Since I is the incenter of ΔABC, we have $\angle AIC = 90° + \frac{1}{2}\angle ABC$ and hence, $\angle AIP = 180° - \angle AIC = 90° - \frac{1}{2}\angle ABC = 90° - \angle CBI$.

Notice that $\angle CBI = \angle PNQ$ (because $MN \,//\, BC$ and $BI \,//\, QN$). Hence, $\angle AIP = 90° - \angle PNQ = \angle PQN$. Since $\angle APC = \angle QPN = 90°$, we must have $\Delta API \sim \Delta NPQ$. Refer to the diagram below.



Now we have $\frac{AP}{PN}=\frac{IP}{PQ}$ and $\angle QPI = \angle APN$.

It follows that $\Delta APN \sim \Delta IPQ$.

Let QI extended intersect AC at D. We have $\angle CID = \angle PIQ = \angle PAC = 90° - \angle ACI$, i.e., $\angle CDI = 90°$. This completes the proof.

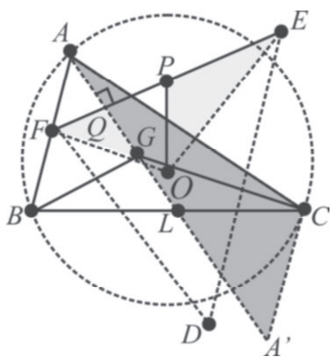
5.Sol: It is easy to see that D, E, F are the circumcenters of $\Delta BCG, \Delta ACG, \Delta ABG$ respectively. Let L, M, N be the midpoints of BC, AC, AB respectively. Notice that the lines DL, EM, FN are the perpendicular bisectors of BC, AC, AB respectively and hence, intersect at O. Let DL extended intersect EF at P. We claim that P is the midpoint of EF.

Let AG intersect EF at Q. Since $AG \perp EF$ and $EM \perp AC, A, E, M, Q$ are concyclic and hence, $\angle CAL = \angle OEP$. (1)

Since $\angle CLO + \angle CMO = 180°$, we also have C, L, O, M concyclic and hence, $\angle ACL = \angle EOP$. (2)

(1) and (2) give $\Delta ACL \sim \Delta EOP$ ane hence, $\frac{EP}{AL} = \frac{OP}{CL}$. (3)

Similarly, one sees that $\Delta ABL \sim \Delta FOP$ and hence, $\frac{FP}{AL} = \frac{OP}{BL}$. (4)



(3) and (4) imply $EP = FP$, i.e., DO extended through the midpoint of EF. Similarly, EO extended and FO extended pass through the midpoints of DF and DE respectively. We conclude that O is the centroid of ΔDEF.

6.Sol: Let O_1, O_2, O_3 denote the centers of $\Gamma_1, \Gamma_2, \Gamma_3$ respectively. Let O_1O_2 intersect AP at M. Clearly, $AM = PM$. Draw $O_3 \perp DP$ at H.

It is easy to see that $DH = PH$. Hence,

$$MH = \frac{1}{2}(AP + DP) = \frac{1}{2}AD.$$

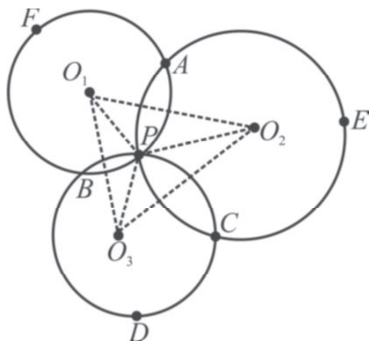
Now $\frac{AP}{AD} = \frac{\frac{1}{2}AP}{\frac{1}{2}AD} = \frac{PM}{HM} = \frac{[\Delta O_1O_2P]}{[\Delta O_1O_2H]}$

Notice that $[\Delta O_1O_2H] = [\Delta O_1O_2O_3] = \frac{1}{2}O_1O_2 \cdot MH$, because $O_1O_2 \perp AD$ and $O_3H \perp AD$, i.e., $O_1O_2 // O_3H$. It follows that $\frac{AP}{AD} = \frac{[\Delta O_1O_2P]}{[\Delta O_1O_2O_3]}$.

Similary, $\frac{BP}{BE} = \frac{[\Delta O_1O_3P]}{[\Delta O_1O_2O_3]}$ and

$$\frac{CP}{CF} = \frac{[\Delta O_2O_3P]}{[\Delta O_1O_2O_3]}.$$

Refer to the diagram below.



The conclusion follows as

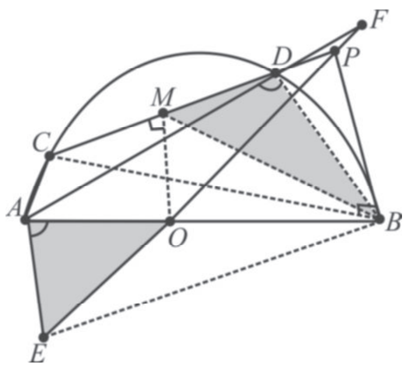
$$\frac{[\Delta O_1O_2P] + [\Delta O_1O_3P] + [\Delta O_2O_3P]}{[\Delta O_1O_2O_3]} = 1$$

7.Sol: Draw $OM \perp CD$ at M. We have $CM = DM$. Since $BP \perp AB$, we have B, O, M, P concyclic and hence, $\angle BMP = \angle BOP = \angle AOE$. (1)
Since A, B, D, C are concyclic, we have $\angle BDC = \angle BAE$. (2)
(1) and (2) imply that $\Delta BDM \sim \Delta EAO$ and hence,

$\frac{AE}{AO} = \frac{BD}{DM}$. Refer to the following diagram.

Since O and M are midpoints of AB, CD respectively, we have

$$\frac{AE}{AB} = \frac{AE}{2AO} = \frac{BD}{2DM} = \frac{BD}{CD} \quad (3)$$

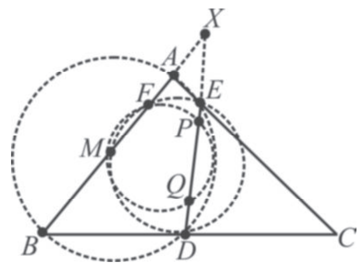


(2) and (3) imply that $\Delta BDC \sim \Delta EAB$. Hence, $\angle BCD = \angle BAD$, we must have $\angle BAD = \angle ABE$. One see that $\Delta AOF \cong \Delta BOE$ (A.A.S.) and hence, $OE = OF$.

8.Sol: Clearly, Q lies on the perpendicular bisector of AB. Let M be the midpoint of AB. We must have $QM \perp AB$. Since $FP \perp DE$, F, M, Q, P are concyclic. Let the lines AB and DE intersect at X. By the Tangent Secant Theorem, $XP \cdot XQ = XF \cdot XM$ (1)

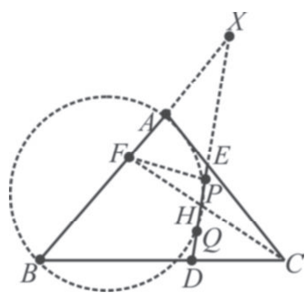
It is well known that A, B, D, E are concyclic and hence, we have $XA \cdot XB = XD \cdot XE$ (2).

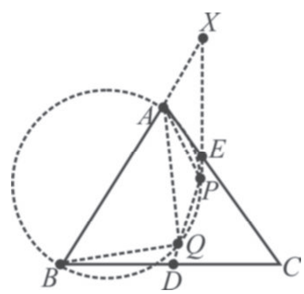
Notice that D, E, F, M are concyclic because they lie on the nine-point circle of ΔABC.
Hence, $XD \cdot XE = XF \cdot XM$ (3).



Refer to the diagram above.
(1), (2) and (3) give $XA \cdot XB = XP \cdot XQ$.

Hence, A, B, Q, P are concyclic and $\angle PAQ = \angle PBQ$.
Let H denote the orthocenter of ΔABC. Consider the right angled triangle ΔFHX. Since $FP \perp HX$, we have $\angle PFC = \angle X$. Refer to the left digram below. If suffices to show $\angle X = \angle PAQ$.





Notice that $\angle X = \angle PAB - \angle APX$, where $\angle APX = \angle ABQ - \angle BAQ$. Ii follows that $\angle X = \angle PAB - \angle BAQ = \angle PAQ$. Refer to the right diagram above. This complete the proof.

CLASS XI MATHEMATICS KVPY-2

PREVIOUS YEAR QUESTIONS

GEOMETRY (PART-1)

(Triangles & Circles)

1. Suppose BC is a given line segment in the plane and T is a scalene triangle. The number of points A in the plane such that the triangle with vertices A,B,C (in some order) is similar to triangle T is
(a) 4 (b) 6 (c) 12 (d) 24

2. Consider four triangles having sides (5,12,9), (5,12,11), (5,12,13) and (5,12,15). Among these, the triangle having maximum area has sides
(a) (5,12,9) (b) (5,12,11)
(c) (5,12,13) (d) (5,12,15)

3. Suppose we have two circles of radius 2 each in the plane such that the distance between their centres is $2\sqrt{3}$. The area of the region common to both circles lies between
(a) 0.5 and 0.6 (b) 0.65 and 0.7
(c) 0.7 and 0.75 (d) 0.8 and 0.9

4. Let C_1, C_2 be two circles touching each other externally at the point A and let AB be the diameter of circle C_1. Draw a secant BA_3 to circle C_2, intersecting circle C_1 at a point $A_1 (\neq A)$, and circle C_2 at points A_2 and A_3. If $BA_1 = 2, BA_2 = 3$ and $BA_3 = 4$, then the radii of circles C_1 and C_2 are respectively

(a) $\frac{\sqrt{30}}{5}, \frac{3\sqrt{30}}{10}$ (b) $\frac{\sqrt{5}}{2}, \frac{7\sqrt{5}}{10}$

(c) $\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}$ (d) $\frac{\sqrt{10}}{3}, \frac{17\sqrt{10}}{30}$

5. The points A, B, C, D, E are marked on the circumference of a circle in clockwise direction such that $\angle ABC = 130°$ and $\angle CDE = 110°$. The measure of $\angle ACE$ in degree is
(a) 50° (b) 60° (c) 70° (d) 80°

6. Three circles of radii 1, 2 and 3 units respectively touch each other externally in the plane. The circumradius of the triangle formed by joining the centres of the circles is
(a) 1.5 (b) 2 (c) 2.5 (d) 3

7. Let P be a point inside a triangle ABC with $\angle ABC = 90°$. Let P_1 and P_2 be the images of P under reflection in AB and BC respectively. The distance between the circumcentres of triangles ABC and $P_1 P P_2$ is

(a) $\frac{AB}{2}$ (b) $\frac{AP+BP+CP}{3}$

(c) $\frac{AC}{2}$ (d) $\frac{AB+BC+AC}{2}$

8. Let a and b be two positive real numbers such that $a + 2b \leq 1$. Let A_1 and A_2 be, respectively, the areas of circles with radii ab and b^2. Then the maximum possible value of $\frac{A_1}{A_2}$ is

(a) $\frac{1}{16}$ (b) $\frac{1}{64}$ (c) $\frac{1}{16\sqrt{2}}$ (d) $\frac{1}{32}$

9. Consider a semicircle of radius 1 unit constructed on the diameter AB, and let O be its centre. Let C be a point on AO such that $AC : CO = 2 : 1$, Draw CD perpendicular to AO with D on the semicircle. Draw OE perpendicular to AD with E on AD. Let OE and CD intersects at H. Then DH equals

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{3}}$

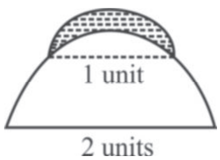
(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{5}-1}{2}$

10. In a triangle ABC, points X and Y are on AB and AC, respectively, such that XY is parallel to BC. Which of the two following equalities always hold? (Here $[PQR]$ denotes the area of triangle PQR)
I. $[BCX] = [BCY]$
II. $[ACX]. [ABY] = [AXY] . [ABC]$
(a) Neither I nor II (b) I only
(c) II only (d) Both I and II only

11. Let P be an interior point of a triangle ABC. Let Q and R be the reflections of P in AB and AC, respectively. If Q, A, R are collinear then $\angle A$ equals
(a) 30° (b) 60° (c) 90° (d) 120°

12. Let $ABCD$ be a square of side length 1, and I^{-} a circle passing through B and C, and touching AD. The radius of I^{-}
(a) $\frac{3}{8}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{5}{8}$

13. A semi-circle of diameter 1 unit sits at the top of a semi-circle of diameter 2 units. The shaded region inside the smaller semi-circle but outside the larger semi-circle is called a lune. The area of the *lune* is

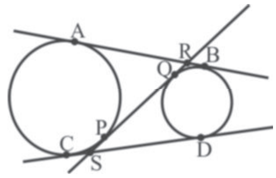


(a) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{4}-\frac{\pi}{24}$

(c) $\frac{\sqrt{3}}{4}-\frac{\pi}{12}$ (d) $\frac{\sqrt{3}}{4}-\frac{\pi}{8}$

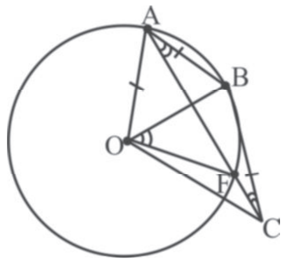
14. The angle bisectors BD and CE of a triangle ABC are divided by the incentre I in the ratios 3 : 2 and 2 : 1 respectively. Then the ratio in which I divides the angle bisector through A is
(a) 3 : 1 (b) 11 : 4 (c) 6 : 5 (d) 7 : 4

15. Suppose S_1 and S_2 are two unequal circles; AB and CD are the direct common tangents to these circles. A transverse common tangent PQ cuts AB in R and CD in S. If $AB = 10$, then RS is



(a) 8 (b) 9 (c) 10 (d) 11

16. On the circle with center O, points A, B are such that $OA = AB$. A point C is located on the tangent at B to the circle such that A and C are on the opposite sides of the line OB and $AB = BC$. The line segment AC intersects the circle again at F. Then the ratio $\angle BOF : \angle BOC$ is equal to



(a) 1 : 2 (b) 2 : 3 (c) 3 : 4 (d) 4 : 5

17. In a triangle ABC with $\angle A = 90°$, P is a point on BC such that $PA : PB = 3 : 4$. If $AB = \sqrt{7}$ and $AC = \sqrt{5}$ then $BP : PC$ is
(a) 2 : 1 (b) 4 : 3 (c) 4 : 5 (c) 8 : 7

18. The numberof values of b for which there is an isosceles triangle with sides of length $b + 5$, $3b - 2$ and $6 - b$ is
(a) 0 (b) 1 (c) 2 (d) 3

19. In a triangle ABC with $\angle A < \angle B < \angle C$, points D, E, F are on the interior of segments BC, CA, AB, respectively. Which of the following triangles **CANNOT** be similar to ABC?
(a) Triangle ABD (b) Triangle BCE
(c) Triangle CAF (d) Triangle DEF

20. Tangents to a circle at points P and Q on the circle intersect at a point R. If $PQ = 6$ and $PR = 5$ then the radius of the circle is
(a) $\frac{13}{3}$ (b) 4 (c) $\frac{15}{4}$ (d) $\frac{16}{5}$

21. In an acute-angled triangle ABC, the altitudes from A, B, C when extended intersect the circumcircle again at points A_1, B_1, C_1, respectively. If $\angle ABC = 45°$ then $\angle A_1B_1C_1$ equals

(a) 45° (b) 60° (c) 90° (d) 135°

22. The sides of a triangle are distinct positive integers in an arithmetic progression. If the smallest side is 10, the number of such triangles is

(a) 8 (b) 9
(c) 10 (d) Infinitely many

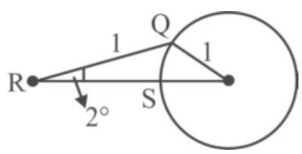
23. In triangle *ABC*, let *AD*, *BE* and *CF* be the intenal angle bisectors with *D*, *E* and *F* on the sides *BC*, *CA* and *AB* respectively. Suppose *AD*, *BE* and *CF* concur at *I* and *B*, *D*, *I*, *F* are concyclic, then $\angle IFD$ has measure

(a) 15° (b) 30°
(c) 45° (d) Any value $\leq 90°$

24. A circle is drawn in a sector of a larger circle of radius *r*, as shown in the adjacent figure. The smaller circle is tangent to the two bounding radii and the arc of the sector. The radius of the small circle is

(a) $\frac{r}{2}$ (b) $\frac{r}{3}$ (c) $\frac{2\sqrt{3}r}{5}$ (d) $\frac{r}{\sqrt{2}}$

25. Suppose *Q* is a point on the circle with centre *P* and radius 1, as shown in the figure; *R* is a point outside the circle such that *QR* = 1 and $\angle QRP = 2°$. Let *S* be the point where the segment *RP* intersects the given circle. Then measure of $\angle RQS$ equals



(a) 86° (b) 87° (c) 88° (d) 89°

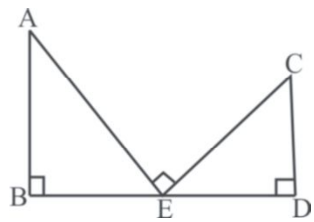
26. In a triangle *ABC*, it is known that $AB = AC$, Suppose *D* is the mid-point of *AC* and $BD = BC = 2$. Then the area of the triangle *ABC* is

(a) 2 (b) $2\sqrt{2}$ (c) $\sqrt{7}$ (d) $2\sqrt{7}$

27. Let *ABC* be a triangle with $\angle B = 90°$. Let *AD* be the bisector of $\angle A$ with *D* on *BC*. Suppose *AC* =6 *cm* and the area of the triangle *ADC* is $10\,cm^2$. Then the length of *BD* in cm is equal to

(a) $\frac{3}{5}$ (b) $\frac{3}{10}$ (c) $\frac{5}{3}$ (d) $\frac{10}{3}$

28. In the adjoining figure $AB = 12\,cm, CD = 8\,cm$, $BD = 20\,cm$; $\angle ABD = \angle AEC = \angle EDC = 90°$. If $BE = x$, then



(a) *x* has two possible values whose difference is 4
(b) *x* has two possible values whose sum is 28
(c) *x* has only one value and $x \geq 12$
(d) *x* cannot be determined with the given information

29. The sides of a triangle *ABC* are positive integers. The smallest side has length 1. Which of the following statement is true?

(a) The area of *ABC* is always a rational number
(b) The area of *ABC* is always an irrational number
(c) The perimeter of *ABC* is an even integer
(d) The information provided is not sufficient to conclude any of the statements *A, B* or *C* above

30. In a triangle *ABC*, *D* and *E* are points on *AB*, *AC* respectively such that *DE* is parallel to *BC*. Suppose *BE*, *CD* intersect at *O*. If the areas of the triangles *ADE* and *ODE* are 3 and 1 respectively, find the area of the triangle *ABC*, with justification

ANSWER KEY

1. c	2. c	3. c	4. a	5. b
6. c	7. c	8. b	9. c	10. d
11. c	12. d	13. b	14. b	15. c
16. b	17. a	18. c	19. a	20. c
21. c	22. b	23. b	24. b	25. d
26. c	27. d	28. a	29. b	30. 12

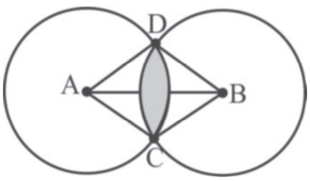
HINTS & SOLUTIONS

1.Sol: We know similar triangles are equiangular. That is corresponding angles are equal.

Given a line segment, where we need to construct a triangle.
That is, we need to choose a pair of angles from 3 angles of another triangle to construct a similar triangle using this line segment. This can be done in $3_{C_2} 2!$ ways. That is, there are 12 similar triangles.

2.Sol: We can easily find the maximum area of the given triangles using Heron's area formula. That is, the triangle (5, 12, 13) has Maximum area.

3.Sol:



From the diagram, we can see that, the area of required curve is 2 times of Area of sector ADC - Area of Rhombus

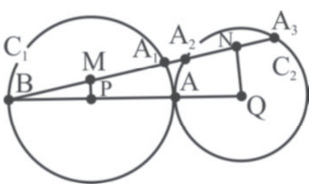
i.e., $2\times\frac{1}{2}\times 2^2\times\frac{\pi}{3}-2\sqrt{3}=\frac{4}{3}\pi-2\sqrt{3}$

$$=\frac{4}{3}\times\frac{22}{7}-2\sqrt{3}$$

$$=4\cdot 190-2(1\cdot 732)$$

$$=0\cdot 726$$

4.Sol:



Given that

$BA_1=2\Rightarrow BM=1$

and $BA_2=3\Rightarrow A_1A_2=1$

and $BA_3=4\Rightarrow A_2N=\frac{1}{2}\Rightarrow BN=\frac{7}{2}$

Let r_1 be the radius of circle $'c_1'$ and r_2 be the radius of circle $'c_2'$.

from the diagram, we see that

ΔQNB and ΔPMB are simillar, since AA corollary.

i.e., $\frac{BM}{BN}=\frac{PM}{QN}$

$$\Rightarrow \quad \frac{PM}{QN}=\frac{1}{7/2}$$

i.e., $\frac{PM}{QN}=\frac{2}{7}$

Now, we have $PM=\sqrt{r_1^2-1}$ and $QN=\sqrt{r_2^2-\frac{1}{4}}$

$$\therefore \quad \sqrt{\frac{r_1^2-1}{r_2^2-\frac{1}{4}}}=\frac{2}{7}$$

i.e., $49r_1^2-49=4r_2^2-1$

$$49r_1^2=4r_2^2+48 \qquad (1)$$

and we have from ΔQNB

$BQ^2=BN^2+NQ^2$

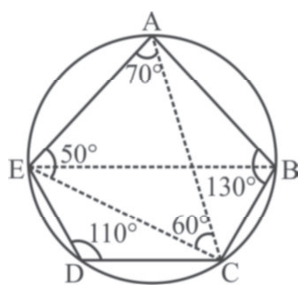
i.e., $(2r_1+r_2)^2=\frac{49}{4}+r_2^2-\frac{1}{4}$

$$\Rightarrow \quad r_1^2+r_1r_2=3 \qquad (2)$$

Solving (1) and (2), we get

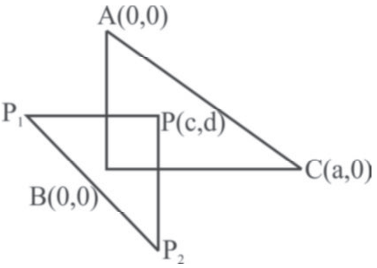
$r_1=\sqrt{\frac{6}{5}}$ and $r_2=\frac{3\sqrt{30}}{10}$

5.Sol:



6.Sol: The traingle in question has side lengths 1 + 2 = 3, 1 + 3 = 4 and 2 + 3 = 5, and is thus a right triangle. The circum radius of a right triangle is one half the length of the hypotenuse, and so in this case is $\frac{1}{2}\times 5=2\cdot 5$.

7.Sol:



M is circumcentre of ΔABC

$\therefore$ Co-ordinate of M is $\left(\frac{a}{2}, \frac{b}{2}\right)$ and N is circumcentre of ΔPP_1P_2

$N = (0,0) = B$ (Mid-point of P_1 and P_2).

$$\therefore \quad MN = \frac{AC}{2}$$

8.Sol: We have

$$A_1 = \pi a^2 b^2$$

and $A_2 = \pi b^4$

$$\Rightarrow \quad \frac{A_1}{A_2} = a^2 b^2$$

Given that $a + 2b \le 1$

i.e., $a + 2b \le 1$

Using AM - GM Inequality,

We get

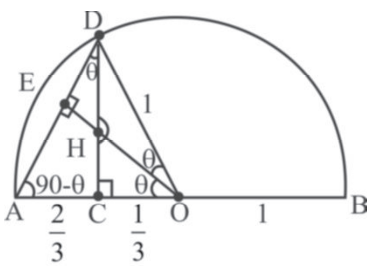
$$\frac{a+2b}{2} \ge (2ab)^{1/2}$$

i.e., $a + 2b \ge 2\sqrt{2ab}$

i.e., $2\sqrt{2ab} \le 1$

$$\Rightarrow a^2 b^2 \le \frac{1}{64}$$

9.Sol:



From $\Delta AOD, E$ is a mid point of AD. Since ΔAOD is isosceles triangle.

now from ΔCDO, we have

$$\cos 2\theta = \frac{1}{3}$$

$$\Rightarrow \quad 2\cos^2\theta - 1 = \frac{1}{3}$$

i.e., $\cos\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$

$$\Rightarrow \quad \sin\theta = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad ED = \sin\theta$$

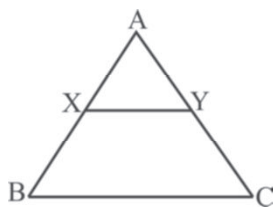
i.e., $ED = \dfrac{1}{\sqrt{3}}$

now from ΔHDE, we have

$$DH = ED \sec\theta$$

i.e., $DH = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

10.Sol:



Clearly ar (BCX) = ar (BCY) $\{\Delta s$ between parallel lines & same bae$\}$

$$\Rightarrow \quad [BCX] = [BCY]$$

(I) is true

(II) $ar(\Delta ACX) = \dfrac{1}{2} AC \cdot AX \sin A$

$$ar(\Delta ABY) = \frac{1}{2} AB \cdot AY \sin A$$

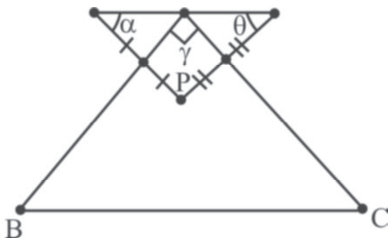
$$ar(\Delta AXY) = \frac{1}{2} AX \cdot AY \sin A$$

$$ar(\Delta ABC) = \frac{1}{2} AB \cdot AC \sin A$$

Clearly $[ACX]\cdot[ABY]=[AXY]\cdot[ABC]$

(II) is true.

11.Sol:



We have

$\angle QAR = 90-\theta+\gamma+90-\alpha$

But $\angle QAR = 180$

i.e., $180-(\alpha+\theta)+\gamma = 180$

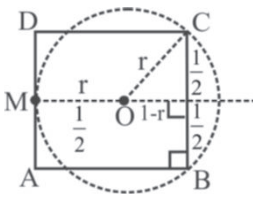
$\Rightarrow \gamma = \alpha+\theta$

But from ΔPQR, we have

$\alpha+\theta = 90^{\circ}$

i.e. $\gamma = 90^{\circ}$

12.Sol:



Let O be centre of circle.

$\therefore \quad OM = \text{radius} = r$

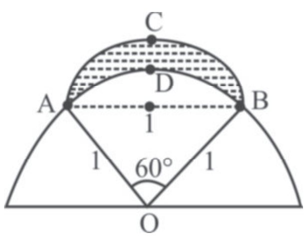
Now, $\quad r^2 = (1-r)^2 + \left(\frac{1}{2}\right)^2$

$\therefore \quad 2r = \frac{5}{4}$

$8r = 5$

$\therefore \quad r = \frac{5}{8}$

13.Sol:



We can see from the diagram, that area of the line is difference between area of semi circle ACB and area of segment ADB

now area of segment ADB = Area of sector ΔADB

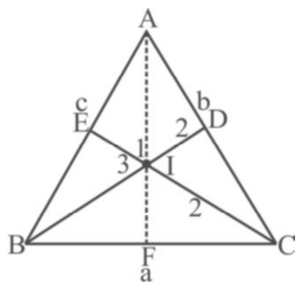
i.e., $= \frac{60^{\circ}}{360^{\circ}}\times\pi(1)^2 - \frac{\sqrt{3}}{4}(1)^2$

$= \frac{\pi}{6}-\frac{\sqrt{3}}{4}$

$\therefore$ desired area is $\frac{\pi}{2}\left(\frac{1}{2}\right)^2 - \left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$

$= \frac{\pi}{8}-\frac{\pi}{6}+\frac{\sqrt{3}}{4}-\frac{\pi}{24}$

14.Sol:



we have, I divides BD in the ratio $c+a:b$ and CE in the ratio $a+b:c$

i.e., $\frac{c+a}{b}=\frac{3}{2}$ *and* $\frac{a+b}{c}=\frac{2}{1}$

$\Rightarrow 2c+2a = 3b$ and $a+b=2c$

i.e., solving above equations, we get $3a = 2b$

$\Rightarrow \frac{\frac{2b}{3}+b}{c} = \frac{2}{1}$

i.e., 5b = 6c

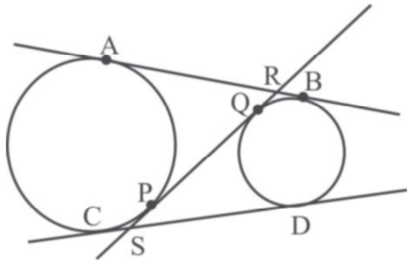
$\therefore a = \frac{2b}{3}$ *and* $c = \frac{5b}{6}$

now, we know I divides AF in the ratio $b+c:a$

i.e., $b+\frac{5b}{6};\frac{2b}{3} \Rightarrow \frac{11b}{6}:\frac{4b}{6}$

i.e., 11:4

15.Sol:



We know $PS = CS = y$ and

$BR = RQ = x$, Since tangents drawn from external point are equal in length.

We know $AR = PQ = 10 - x$ and $PQ = 10 - 2x$

$$\text{Now } CD = CS + SD = y + SP + PQ$$

$$10 = y + y + 10 - 2x$$

$$\Rightarrow \quad y = x$$

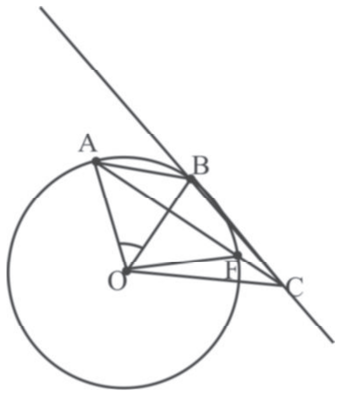
$$\therefore \quad RS = SP + PQ + QR$$

$$= y + 10 - 2x + x$$

$$= 10 + y - x$$

$$= 10$$

16.Sol:



Given OA = AB

$\Rightarrow OB = AB$

$\therefore \quad \Delta OAB$ is equilateral

We have ΔABC we have

$\angle BAC + \angle ABC + \angle BAC = 180$

i.e., $150 + \angle BCA + \angle BCA = 180$

$\angle BCA = 30$

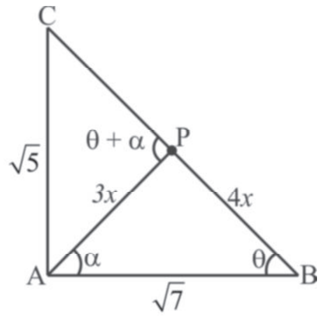
$\angle BCA = 15°$

We know ΔOBC is isosceles [Since OB = BC]

$$\angle BOC = 45° \text{ and } \angle BOF = 30°$$

$$\text{Now } \frac{\angle BOF}{\angle BOC} = \frac{30°}{45°} = \frac{2}{3}$$

17.Sol:



Given that $PA = 3x$ and $PB = 4x$

Let $\theta = \angle ABC$ and $\angle PAB = \alpha$

Then

$$\tan\theta = \frac{\sqrt{5}}{\sqrt{7}} \Rightarrow \sin\theta = \frac{\sqrt{5}}{\sqrt{12}};$$

Now, we have sine rule in $\Delta\, APB$

$$\text{i.e.,} \quad \frac{3x}{\sin\theta} = \frac{4x}{\sin\alpha} = \frac{\sqrt{7}}{\sin(180° - (\theta + \alpha))}$$

$$\Rightarrow \sin\alpha = \frac{4}{3}\sin\theta = \frac{4}{3} \times \frac{\sqrt{5}}{\sqrt{12}} = \frac{\sqrt{20}}{\sqrt{27}}$$

$$\therefore \quad \cos\alpha = \frac{\sqrt{7}}{\sqrt{27}}$$

Now from ΔAPB, we have sine rule

$$\text{i.e.,} \quad \frac{3x}{\sin\theta} = \frac{\sqrt{7}}{\sin(\theta + \alpha)}$$

$$\text{and} \quad 3x\sin(\theta + \alpha) = \sqrt{7}\sin\theta$$

$$\text{i.e.,} \quad 3x\left(\frac{\sqrt{7}}{\sqrt{27}} + \frac{\sqrt{7}}{\sqrt{6}}\frac{\sqrt{20}}{\sqrt{27}}\right) = \sqrt{7}$$

$$\therefore \quad 3x \times \frac{1}{\sqrt{3}} = 1$$

$$\therefore \quad x = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{BP}{PC} = \frac{4x}{\sqrt{12-4x}}$$

$$= \frac{\frac{4}{\sqrt{3}}}{\sqrt{12-\frac{4}{\sqrt{3}}}} = \frac{2}{1} = 2$$

18.Sol: The triangle in which we have lengths of any two sides equal is called are isosceles triangle. So here we are given length of an isosceles triangle $b+5, 3b-2, 6-b$

The possible equal sides are either $b+5$ *and* $3b-2$ or $3b-2$ *and* $6-b$ or $b+5$ *and* $6-b$

Case (I)

The possible isosceles triangle with equal side lengths i.e. $b+5=3b-2$

$$\Rightarrow \qquad b = \frac{7}{2}$$

Now the length od sides of the triangles are 8.5, 8.5, 2.5. So triangle is possible with these measurements

Case (II)

The possible isosecles triangle with equal sides lengths of $3b-2$ *and* $6-b$

i.e. $\qquad 3b-2=6-b$

i.e. $\qquad b=2$

Noe the length of sides of the triangle are 7, 4, 4. So triangle is possible with these measurements

Case (III)

The possible isosceles triangle with equal side length of $b+5$ *and* $6-b$

i.e. $\qquad 3b-2=6-b$

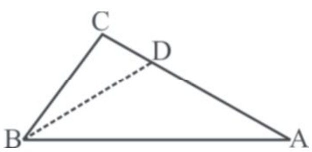
$$b+5=6-b \Rightarrow b = \frac{1}{2} = 0.5$$

Now the length of sides of the triangle 5.5, -0.5, 5.5, which is not possible as the length cannot be negative.

Hence possible values of b are 3,5,2.

These are 2 possible values.

19.Sol:

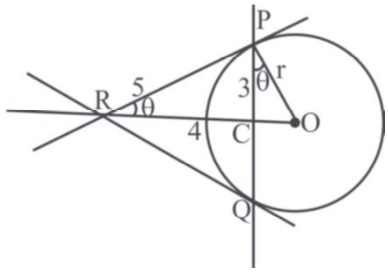


We have $\angle BDA = \angle DBC + \angle C$

$$\Rightarrow \angle BDA > \angle C$$

$\therefore \Delta ABD$ cannot be Similar to ΔABC

20.Sol:



We have ΔRPC *and* ΔPOC are Simillar. That is

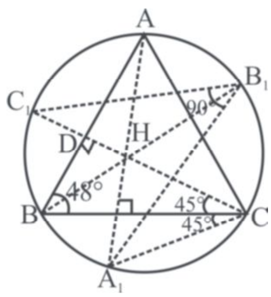
$$\frac{RP}{PO} = \frac{PC}{OC} = \frac{RC}{PC}$$

$$\Rightarrow \frac{RP}{PO} = \frac{RC}{PC}$$

i.e. $$\frac{5}{r} = \frac{4}{3}$$

$$\Rightarrow r = \frac{15}{4}$$

21.Sol:



We have $AD \perp BC, BE \perp AC$ *and* $CF \perp AB$

and $\angle ABC = 45^\circ$

In ΔABD

$$\angle BAD + \angle ABD + \angle ADB = 180^\circ$$

$$\Rightarrow \angle BAD = 180^\circ - 135^\circ$$

i.e., $\angle BAD = 45^\circ$

$$\therefore \angle BAD = \angle BAA_1 = 45^\circ \qquad (1)$$

Also, in ΔBCF

$\angle BCF + \angle BFC + \angle CBF = 180^\circ$

i.e., $\angle BCF = 180^\circ - 135^\circ$

$= 45^\circ$

$\therefore \angle BCF = \angle BCC_1 = 45^\circ$ (2)

As, $\angle BB_1A_1 = \angle BAA_1$ [Angles on the same arc are equal]

$\Rightarrow \angle BB_1A_1 = 45^\circ$

Also $\angle BB_1C_1 = \angle BCC_1$

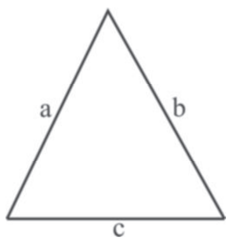
$\angle BB_1A_1 = 45^\circ$

Now, $\angle A_1B_1C_1 = \angle BB_1A_1 + \angle BB_1C_1$

$= 45^\circ + 45^\circ$

$\therefore \angle A_1B_1C_1 = 90^\circ$

22.Sol:



Given a, b, c are in AP (criteria to form any triangle)

$\Rightarrow$ $10, 10+d, 10+2d$

using triangular inequality, we get

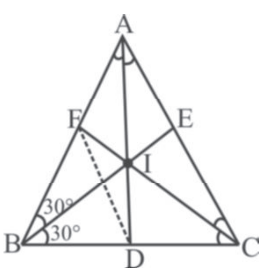
$a + b > c$

$\therefore$ $20 + d > 10 + 2d$

$\therefore$ $10 > d$

As the d is minimum, Hence total possibility of d is 9

23.Sol:



$$\angle AIC = 180° - \left(\frac{A+C}{2}\right)$$

Given that BDIF is concyclec

i.e. $\angle B + \angle DIF = 180$

$\Rightarrow \angle DIF = \angle 180 - \angle B$

and we know, vertically opposite angle are equal. that is

$\angle DIF = \angle AIC$

$$180 - \angle B = 90 + \frac{\angle B}{2}$$

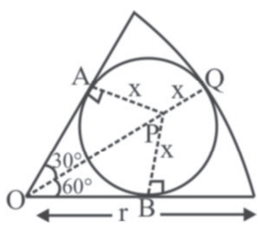
$$\Rightarrow 3\frac{\angle B}{2} = 90$$

i.e. $\Rightarrow 3\dfrac{\angle B}{2} = 90$

$\therefore$ This will be case of equilateral triangle

$\therefore \angle IFD = 30^\circ$

24.Sol:



Let the radius of smaller circle be x

Here,

In ΔOAP

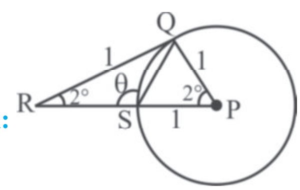
we have $\operatorname{cosec} 30° = \dfrac{OP}{x}$

$\therefore$ $OP = x \operatorname{cosec} 30°$

and $OQ = r = x + x \operatorname{cosec} 30°$

$\therefore$ $x = \dfrac{r}{3}$

25.Sol:



using Cosine rule, we have

$(QS)^2 = 2 - 2\cos 2°$

$\therefore \quad QS = 2\sin 1°$

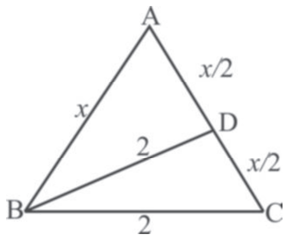
Now by sine rule in ΔRQS ,We have

$$\therefore \quad \sin\theta = \frac{\sin 2°}{2\sin 1°}$$

$$\therefore \quad \theta = 89°$$

$$\therefore \quad \angle RQS = 180° - (2° + 89°) = 89°$$

26.Sol:



Given that

$AB = BC; AD = DC \text{ and } BD = BC = 2$

We have Apollonius theorem,

i.e. $AB^2 + BC^2 = 2\left(CD^2 + BD^2\right)$

$$\Rightarrow x^2 + 4 = 2\left(\frac{x^2}{4} + 4\right)$$

i.e., $x^2 - \frac{x^2}{2} = 8 - 4$

$$\Rightarrow \frac{x^2}{2} = 4$$

$\therefore x = \sqrt{8}$

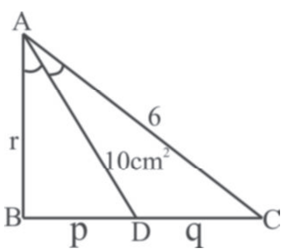
Now area of ΔABC is

$\sqrt{s(s-a)(s-b)(s-c)}$

i.e., $\sqrt{\left(\sqrt{8}+1\right)(1)(1)\left(\sqrt{8}-1\right)}$

$= \sqrt{8-1} = \sqrt{7}$ Sq.unit

27. Sol:



Let P be the length of BD.

We have angle bisector theorem

i.e. $\frac{\gamma}{6} = \frac{P}{2}$

$$\Rightarrow \quad qr = 6p \qquad (1)$$

Now Area of $\Delta ADC = \frac{1}{2}(DC)(AB)$

$$\therefore \quad 10 = \frac{1}{2}(q)\,(r)$$

$$\therefore \quad qr = 20 \qquad (2)$$

Solving (1) and (2) ,

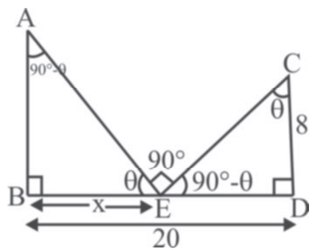
$$20 = 6p$$

$$\therefore \quad p = \frac{20}{6} = \frac{10}{3}$$

28.Sol:

Given that AB = 12 cm, CD = 8 cm, CD = 8 cm, BD = 2 cm, and

$\angle ABD = \angle AEC = \angle EDC = 90^{\circ}$



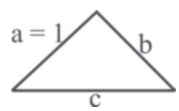
We have $\Delta ABC \cong \Delta EDC$

i.e. $\frac{12}{x} = \frac{20-x}{8}$

$$\Rightarrow \quad x^2 = 20x + 96 = 0$$

$$\therefore \quad x = 8, 12$$

29.Sol:



Let the smallest side be $a = 1$

Using triangular inequality,

we get

$1 + b > c \Rightarrow b - c > -1$

and

$1+c>b \Rightarrow b-c<1$

$\therefore \quad -1<b-c<1$

given that b, c are integers so $b-c=0 \Rightarrow b=c$

Now, semiperimeter (s) $=\dfrac{2b+1}{2}=b+\dfrac{1}{2}$

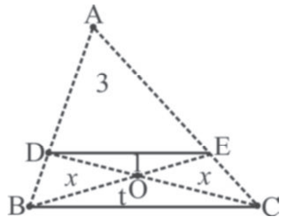
$\therefore$ Area

$$=\sqrt{\left(b+\frac{1}{2}\right)\left(b+\frac{1}{2}-b\right)\left(b+\frac{1}{2}-c\right)\left(b+\frac{1}{2}-1\right)}$$

$$=\frac{1}{2}\sqrt{b^2-\frac{1}{4}}=is\text{ Irrational}$$

30.Sol: We denote the area of triangle PQR by $[PQR]$. We see that $[BOD]$ and $[COE]$ are equal. Let the common value be x, and let $[BOC]=t$. Using the fact that the ratio of areas of two triangles having equal altitudes is the same as the ratio of their respective bases, we obtain.

$$\frac{x}{1}=\frac{BO}{OE}=\frac{t}{x}$$



This gives $t=x^2$. Now ADE and ABC are similar so that

$$\frac{[ADE]}{[ABC]}=\frac{DE^2}{BC^2}=\frac{[ODE]}{[OBC]}$$

since ODE and OCB are also similar. This implies that

$$\frac{3}{4+2x+t}=\frac{1}{t}$$

which simplifies to $t=2+x$, using $t=x^2$ we get a quadratic in $x: x^2-x-2=0$. Its solution are $x=2$ and $x=-1$. Since x cannot be negative, $x=2$ and $t=4$. Thus $[ABC]=4+2x+t=4+4+4=12$.

Synopticglance

DETERMINANTS

Introduction

Both determinants and matrices were introduced as **mathematical short hand** while studying system of linear equalities, let $a_1x+b_1y=0$ and $a_2x+b_2y=0$ be the two homogeneous linear equations.

Multiplying the first equation by b_2, the second by b_1, substracting and dividing by x, then $a_1b_2-a_2b_1=0$ which is some times written as

$$\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0$$

The expression on the left is called **determinant** and it is denoted by Δ. This is **second order** because it has two rows and two columns. The letter a_1, b_1, a_2, b_2 are called the **elements** of the determinant. The value of

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1.$$

A determinant which consists of 3 rows and 3 columns is called **third order** and the

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=(a-b)(b-c)(c-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$$

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$=a_1\left(b_2c_3-b_3c_2\right)-b_1\left(a_2c_3-a_3c_2\right)+c_1\left(a_2b_3-a_3b_3\right)$$

IMPORTANT POINTS

- A determinant can be expand with respect to any row(column), the value will be the same.

Minor & Cofactor of an Element of a Determinant

Let $A=[a_{ij}]$ be a square matrix, then

- The minor of the a_{ij} of $|A|$ is the value of the determinant obtained by deleting its i^{th} row and j^{th} column and it is denoted by M_{ij}
- The cofactor of the element a_{ij} of $|A|$ is denoted by the corresponding capital letter C_{ij} and $C_{ij}=(-1)^{i+1}M_{ij}$.

Properties of Determinants

- $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

  i.e., $\det(A^T)=\det A$
- If you change two rows (columns) of a matrix you reverse the sign of its determinant from positive to negative or from negative to positive.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

- If A has a row (column) that is called zeros, then $\det A = 0$.

$$A = \begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 0 \end{vmatrix}$$

- If two rows (columns) of a matrix are equal, its determinant is zero.

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

- The determinant behaves like a linear function on the rows (columns)

$$\begin{vmatrix} a_1+x_1 & b_1 & c_1 \\ a_2+x_2 & b_2 & c_2 \\ a_3+x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$$

- If we multiply one row of a matrix by p, the determinant is multiplied by p.

$$\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ pa_2 & pb_2 & pc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{vmatrix} = p^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

IMPORTANT POINTS

- $\begin{bmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{bmatrix} = p\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$
- If the order of A is n, then $\det(\lambda A) = \lambda^n \det(A)$

- The determinant of a triangular matrix is the product of the diagonal entries $d_1, d_2, ..., d_n$.
- The determinant of a permutation matrix A is 1 or depending on whether A exchanges an even or odd number of rows (columns).
- $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\lambda b_1 & b_1 & c_1 \\ a_2+\lambda b_2 & b_2 & c_2 \\ a_3+\lambda b_3 & b_3 & c_3 \end{vmatrix}$

IMPORTANT POINTS

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \xrightarrow{\alpha C_2+\beta C_3+C_1}$$

$$\begin{bmatrix} x_1+\alpha y_1+\beta z_1 & y_1 & z_1 \\ x_2+\alpha y_2+\beta z_2 & y_2 & z_2 \\ x_3+\alpha y_3+\beta z_3 & y_3 & z_3 \end{bmatrix}$$

- If any line of a determinant Δ be passed over p parallel lines, the resultant determinant is $(-1)^p\Delta$.
- When the elements of a determinant Δ are rational integral functions of x (polynomials) and two rows or columns become identical when $x = a$, then $(x-a)$ is a factor of Δ. If r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of Δ.
- Differentiation of determinant $\Delta = \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$

where f_r, g_r, h_r are functions of x for $r = 1,2,3$.

$$\therefore \frac{d\Delta}{dx} = \begin{vmatrix} f'_1 & g'_1 & h'_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & h_1 \\ f'_2 & g'_2 & h'_2 \\ f_3 & g_3 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f'_3 & g'_3 & h'_3 \end{vmatrix}$$

Optimal value of Determinants when Elements are known

If

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \text{ where } a_i\text{'s} \in \{\alpha_1, \alpha_2, ..., \alpha_n\},$$

then $|A|_{\max}$ when diagonal elements are $\min\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and non-diagonal elements are $\max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ and also $|A|_{\min} = -|A|_{\max}$.

Multiplication of Determinants

Definition:

Let $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, |B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$. Then

$$|A||B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

Theorem

(1) $\det(AB) = (\det A)(\det B)$ i.e., $|AB| = |A||B|$

(2) $|A|(|BC|) = (|AB|)|C|$; N.B

; $A(BC) = (AB)C$

(3) $|A||B| = |B||A|$; N.B; $AB \neq BA$ in general

(4) $|A|(|B| + |C|) = |A||B| + |A||C|$

Definition: Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then A_{ij}, the cofactor of a_{ij}, is defined by

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, ..., C_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Since

$$|A| = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{21}C_{21} - a_{22}C_{22} + a_{23}C_{23}$$

Theorem

(1) $a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3} = \begin{cases} \det A & if\ i = j \\ 0 & if\ i \neq j \end{cases}$

(2) $a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = \begin{cases} \det A & if\ i = j \\ 0 & if\ i \neq j \end{cases}$

Inverse of Square Matrix by Determinants

Definition: The cofactor matrix of A is defined as

$$cofA = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

Definition: The adjoint matrix of A is defined as

$$adjA = (cofA)^T = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{21} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Theorem: For any square matrix A of order n.

$$A(adj\ A) = (adj\ A)A = (\det A)I$$

$$A(adj\ A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

Theorem: Let A be a square matrix. If $\det A \neq 0$, then A is non - singular and $A^{-1} = \frac{1}{\det A}(adjA)$.

Theorem: A square matrix A is non-singular iff $\det A \neq 0$.

IMPORTANT POINTS

- A is singular (non-invertible) iff A^{-1} does not exist.
  Theorem: A square matrix A is singular iff $\det A = 0$.

Properties of inverse matrix

Let A, B be two non - singular matrices of the same order and λ be a scalar.

- $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^{T})^{-1} = (A^{-1})^{T}$
- $(A^{n})^{-1} = (A^{-1})^{n}$ for any positive integer n.
- $(AB)^{-1} = B^{-1}A^{-1}$
- The inverse of a square matrix is unique.
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$

IMPORTANT POINTS

- $XY = 0 \not\Rightarrow X = 0$ or $Y = 0$

  $\Rightarrow A^{-1}(AX) = 0$

- If A is non - singular, then

$$AX = 0 \Rightarrow A^{-1}(AX) = AO \Rightarrow I(X) = O \Rightarrow X = O$$

IMPORTANT POINTS

- $XY = XZ \not\Rightarrow X = 0$ or $Y = Z$
- $AX = AY \Rightarrow A^{-1}AX = A^{-1}AY$
- If A is non-singular, then

$$AX = AY = A^{-1}AX = A^{-1}AY \Rightarrow X = Y$$

- $(A^{-1}MA)^{n} = (A^{-1}MA)(A^{-1}MA)(A^{-1}MA)\cdots$

$$(A^{-1}MA)......(A^{-1}M^{n}A)$$

- If $M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then

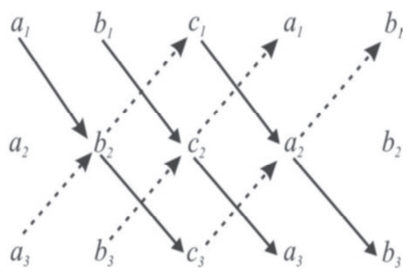
$$M^{-1} = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$$

- If $M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then $M^{n} = \begin{pmatrix} a^{n} & 0 & 0 \\ 0 & b^{n} & 0 \\ 0 & 0 & c^{n} \end{pmatrix}$,

  where $n \neq 0$.

Sarrus Rule for Expansion

Sarrus gave a rule for a determinant of order 3.
Rule : The three diagonals sloping down to the right give the three positive terms, and the three diagonals sloping down to the left give the three negative terms.



$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3$$

$$-a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Some Operations

The first, second and third rows of a determinant are denoted by R_1, R_2 and R_3 respectively, and the first, second and third columns by C_1, C_2 and C_3, respectively.

Properties

- The interchange of its i^{th} row and j^{th} row is denoted by $R_i \to R_j$.

- The interchange of i^{th} column and j^{th} column is denoted by $C_i \leftrightarrow C_j$.
- The addition of m-times the elements of j^{th} row of the corresponding elements of i^{th} row is denoted by $R_i \to R_i + mR_j$.
- The addition of m-times the elements of j^{th} column to the corresponding elements of i^{th} column is denoted by $C_i \to C_i + mC_j$.
- The addition of m-times the elements of j^{th} row to n-times the elements of i^{th} row is denoted by
$$R_i \to nR_i + mR_j.$$

Applications

(1) Use of determinant in coordinate geometry

Area of Triangle

The area of a triangle, whose vertices are

$(x_1, y_1), (x_2, y_2)$, and (x_3, y_3), is

$$\frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

(2) Condition of concurrency of three lines

Three lines are said to be concurrent if they pass through a common point, i.e., they meet at a point.

Let $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$; be three concurrent lines,

then
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Condition for General Second Degree Equation in x and y represent pair of straight lines. The general second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(3) Determinant of characteristic roots and vector

If λ is a characteristic root and X is a corresponding characteristic vector of a matrix A, then we have

$$AX = \lambda X = \lambda IX \text{ or } (A - \lambda I)X = 0$$

Since $X \neq 0$, we deduce that the matrix $(A - \lambda I)$ is singular so that its determinant

$$|A - \lambda I| = 0$$

Thus, every characteristic root λ of a matrix A is root of its characteristic equation

$$|A - \lambda I| = 0 \qquad \text{... (1)}$$

Conversely, if λ is any root of the characteristic equation [Eq. (1)], then the matrix equation $(A - \lambda I)X = 0$ necessarily possesses a nonzero solution X so that there exists a vector $X \neq 0$ such that $AX = \lambda IX = \lambda X$.

Thus, every root of the characteristic equation of a matrix is a characteristic root of the matrix.

If A is n-rowed, then the characteristic equation $|A - \lambda I| = 0$ is of n[th] degree so that every n-rowed square matrix possesses n characteristic roots, which, of course, may not all be distinct.

Cyclic order:

$$f(a,b,c) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \text{ then}$$

$$f(a,a,c) = f(a,b,b) = f(c,b,c) = 0$$

then $a-b, b-c, c-a$ are factors of determinant.

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Exercise

1. While expanding the determinant instead of multiplying by $(-1)^{i+j}$, we can multiply by $+1$ or -1 according as $i+j$ is
(a) Odd (b) Even or odd
(c) Odd or even (d) Even

2. When the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expand in powers of $\sin x$, then the constant term in that expression is
(a) 2 (b) 1 (c) -1 (d) 0

3. The determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is
(a) Independent of θ
(b) Independent of both θ and x
(c) Independent of x only
(d) None of these

4. If $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, then the cofactor A_{21} is
(a) $-(hc+fg)$ (b) $fg-hc$
(c) $fg+hc$ (d) $hc-fg$

5. If $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, the values of x and y are respectively
(a) Δ_2/Δ_1 and Δ_3/Δ_1 (b) Δ_1/Δ_3 and Δ_2/Δ_3
(c) Δ_1/Δ_3 and Δ_2/Δ_3 (d) $e^{\frac{\Delta_1}{\Delta_3}}$ and $e^{\frac{\Delta_2}{\Delta_3}}$

6. The minors of -4 and 9 and the cofactor of -4 and 9 in the determinant $\begin{vmatrix} -1 & -2 & 3 \\ -4 & -5 & -6 \\ -7 & 8 & 9 \end{vmatrix}$ are respectively
(a) 42, 3 ; 42; 3 (b) 42, 3 ; -42, 3
(c) 42, 3; -42, -3 (d) -42, -3 ; 42,-3

7. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, then $f(2x)-f(x)$ equals
(a) $x(2a+3x)$ (b) $a(2a+3x)$
(c) $ax(2a+3x)$ (d) $ax(2x+3a)$

8. If $f(\theta) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & e^{i\theta} & 1 \\ 1 & -1 & -e^{-i\theta} \end{vmatrix}$, then
(a) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\,d\theta = 2\int_{0}^{\frac{\pi}{2}} f(\theta)\,d\theta$ (b) $f\left(\frac{\pi}{2}\right)=0$
(c) $f(\theta)$ is purely imaginary
(d) None of these

9. If $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = k\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$, then k equal to
(a) 1 (b) -2 (c) -1 (d) 2

10. The value of the determinant
$\begin{vmatrix} 1 & 4 & 7 & 10 \\ 4 & 7 & 10 & 13 \\ 7 & 10 & 13 & 16 \\ -2 & 3 & 1 & 0 \end{vmatrix}$ is equal to
(a) 10 (b) 22 (c) 0 (d) None of these

11. If a determinant of order 3×3 is formed by using the numbers 1 or -1, then the minimum value of the determinant is
(a) -8 (b) -4 (c) 0 (d) -2

12. Let the determinant of a 3×3 matrix A be 6 and B is a matrix defined by $B=5A^2$, then det B is equal to
(a) 100 (b) 80 (c) 180 (d) None of these

13. $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix} =$
(a) 7 (b) 10 (c) 13 (d) 17

14. If $\begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix}$ then the value of

$\int_{-\pi/2}^{\pi/2} f(x)dx$ is equal to

(a) 2 (b) 1
(c) 3 (d) None of these

15. If A is a 3×3 non-singular matrix, then $|A^{-1}adjA|$ is

(a) $|A|$ (b) $|A|^{-1}$ (c) 1 (d) $|A|^2$

ANSWER KEY

1. b	**2.** c	**3.** a	**4.** b	**5.** d
6. d	**7.** c	**8.** a	**9.** a	**10.** c
11. b	**12.** d	**13.** b	**14.** d	**15.** a

HINTS & SOLUTIONS

1.Sol: While expanding the determinant instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or -1 according as $(i+j)$ is even or odd.

2.Sol: Let $f(x)=\begin{vmatrix} \cos(2x) & \sin^2(x) & \cos(4x) \\ \sin^2(x) & \cos(2x) & \cos^2(x) \\ \cos(4x) & \cos^2(x) & \cos(2x) \end{vmatrix}$

We know, to find constant term of any polynomial function, we need to put $x=0$

i.e., $\sin x = 0 \Rightarrow x = 0$

$$\therefore \ f(0)=\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$=1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

3.Sol: Given $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$$= x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$= x(-x^2-1) - \sin\theta(-\sin\theta x - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$$

$$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) + 0 = -x^3$$

4.Sol: We have, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Now minor of a_{21} is $M_{21} = \begin{vmatrix} h & g \\ f & c \end{vmatrix} = hc - fg$

$\therefore \ A_{21} = (-1)^{2+1} \ M_{21} = -(hc - fg) = fg - hc$

5.Sol: Given $x^a y^b = e^m$ and $x^c y^d = e^n$

$\Rightarrow a\log x + b\log y = m$ and

$c\log x + d\log y = n$

Using Cramer's rule, we have

$$\log x = \frac{\Delta_1}{\Delta_3}; \quad \log y = \frac{\Delta_2}{\Delta_3}$$

$$\therefore \ x = e^{\frac{\Delta_1}{\Delta_3}} \text{ and } y = e^{\frac{\Delta_2}{\Delta_3}}$$

6.Sol: Minor of -4 is $\begin{vmatrix} -2 & 3 \\ 8 & 9 \end{vmatrix} = -42,$

likewise minor of 9 is $\begin{vmatrix} -1 & -2 \\ -4 & -5 \end{vmatrix} = -3$

and cofactor of -4 is $(-1)^{2+1}(-42) = 42$

and that of 9 is $(-1)^{3+3}(-3) = -3.$

7.Sol: Applying $R_2 \to R_2 - xR_1$ and $R_3 \to xR_2$,

We get $f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & (a+x) & -1 \\ 0 & 0 & a+x \end{vmatrix}$

$$= a(a+x)^2$$

$$\therefore \ f(2x) - f(x) = a(a+2x)^2 - a(a+x)^2$$

$$= ax(2a+3x)$$

8.Sol: On operating $R_1 - R_2 \Rightarrow R_1$ and

$R_3 - R_2 \Rightarrow R_3$

$$f(\theta) = \begin{vmatrix} 0 & 1-e^{i\theta} & -2 \\ 1 & e^{i\theta} & 1 \\ 0 & -1-e^{-i\theta} & -1-e^{-i\theta} \end{vmatrix}$$

$$= (-1)\left[\left(1-e^{i\theta}\right)\left(-1-e^{-i\theta}\right) - 2\left(-1-e^{-i\theta}\right)\right]$$

$$= 2(1+\cos\theta)$$

Now $f(-\theta) = f(\theta) \Rightarrow f(\theta)$ is an even function.

9.Sol: Given $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = k \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Let $\Delta = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

$aR_1 \to R_1$; $bR_2 \to R_2$ and $cR_3 \to R_3$

we get $\Delta = \dfrac{1}{abc} \begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$

$$= \frac{(abc)}{abc} \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \Rightarrow \Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$\therefore \quad k = 1$

10.Sol: Given $\begin{vmatrix} 1 & 4 & 7 & 10 \\ 4 & 7 & 10 & 13 \\ 7 & 10 & 13 & 16 \\ -2 & 3 & 1 & 0 \end{vmatrix}$

$R_2 - R_1 \to R_2$ and $R_3 - R_2 \to R_3$

$$= \begin{vmatrix} 1 & 4 & 7 & 10 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ -2 & 3 & 1 & 0 \end{vmatrix} = 0$$

11.Sol: Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{vmatrix}$

Applying $C_2 \to C_2 - \dfrac{a_{12}}{a_{11}} C_1, C_3 - \dfrac{a_{13}}{a_{11}} C_1$, we get

$$D = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \dfrac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \dfrac{a_{13}}{a_{11}} a_{21}\right) \\ a_{31} & \left(a_{32} - \dfrac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{33} - \dfrac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$

which has minimum value of -4.

12.Sol: Given $|A| = 6$ also given

$$|B| = |5A^2| = 5^3 |A|^2 = 125(36) = 4500$$

13.Sol: Given $\begin{vmatrix} \log_3^{512} & \log_4^3 \\ \log_3^8 & \log_4^9 \end{vmatrix} \times \begin{vmatrix} \log_2^3 & \log_8^3 \\ \log_3^4 & \log_3^4 \end{vmatrix}$

$$\Rightarrow \left[\log_3^{512} \log_4^9 - \log_3^8 \log_4^3\right]\left[\log_3^4 \log_2^3 - \log_3^4 \log_8^3\right]$$

$$\left(9 - \frac{3}{2}\right)\left(2 - \frac{2}{3}\right) = 10$$

14.Sol: Let $f(x) = \begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix}$

$$f(-x) = \begin{vmatrix} -x & \cos(-x) & e^{(-x)^2} \\ -\sin x & x^2 & \sec x \\ -\tan x & 1 & 2 \end{vmatrix} = -f(x)$$

Now $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx = 0$

15.Sol: $|A^{-1} adjA| = |A^{-1}||adjA| = |A|^{-1}|A|^2 = |A|$

$$\left\{\because |A^{-1}| = |A|^{-1}\right\}$$